



“Kepler Harmonies” and conformal symmetries

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ABSTRACT

Kepler’s rescaling becomes, when “Eisenhart-Duval lifted” to 5-dimensional “Bargmann” gravitational wave spacetime, an ordinary spacetime symmetry for motion along null geodesics, which are the lifts of Keplerian trajectories. The lifted rescaling generates a well-behaved conserved Noether charge upstairs, which takes an unconventional form when expressed in conventional terms. This conserved quantity seems to have escaped attention so far. Applications include the Virial Theorem and also Kepler’s Third Law. The lifted Kepler rescaling is a Chrono-Projective transformation. The results extend to celestial mechanics and Newtonian Cosmology.

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1. Introduction

In today’s language, Kepler’s Third Law of planetary motion [1] states that the planetary trajectories are taken into themselves under the rescaling

$$t \rightarrow \Lambda^3 t, \quad \mathbf{x} \rightarrow \Lambda^2 \mathbf{x}, \quad \Lambda = \text{const.} \quad (1.1)$$

where t denotes non-relativistic time and \mathbf{x} the planet’s position. An intriguing feature is that the standard Lagrangian in 3 + 1 non-relativistic dimensions changes under (1.1) as,

$$\mathcal{L}_{\text{Kepler}} = \frac{m}{2} \left(\frac{d\mathbf{x}}{dt} \right)^2 + \frac{GmM_{\odot}}{|\mathbf{x}|} \rightarrow \Lambda^{-2} \mathcal{L}_{\text{Kepler}}, \quad (1.2)$$

and is therefore *not a symmetry* in the sense that the Lagrangian does not change by a total derivative; some textbooks call it a “similitude” [2].

The aim of this Note is to celebrate the 400 years of Kepler’s “*Harmonices Mundi*” which first stated the Third Law [1] by providing new insight by “Eisenhart-Duval lifting” the problem to “Bargmann” space, which is in fact the space-time of a plane

gravitational wave in 5-dimensions [3–6]. The classical motions downstairs are the projections from the Bargmann space of *null geodesics*.

The clue is then that Kepler’s rescaling is a *Chrono-Projective transformation* [7] which becomes, when lifted to “Bargmann space”, a particular type of conformal isometry [5,8], which acts as a perfectly well behaving symmetry for null geodesics “upstairs” and provides us with a perfectly well behaved conserved quantity however when expressed in terms of the original non-relativistic variables “downstairs”, this quantity takes an unconventional form.

2. Kepler rescaling

Let us recall that the Bargmann manifold of a $(d, 1)$ dimensional non-relativistic system is a $d + 2$ dimensional manifold \mathcal{M} endowed with a metric with Lorentz signature which also carries a covariantly constant null vector ξ which we call here the “vertical vector”. In the case we are interested $d = 3$; in suitable coordinates $\mathbf{x} \in \mathbb{R}^3$, $t, s \in \mathbb{R}$, the metric and the vertical vector are,

$$g_{\mu\nu} dx^{\mu} dx^{\nu} = d\mathbf{x}^2 + 2dt ds + \frac{2GM_{\odot}}{|\mathbf{x}|} dt^2 \quad \text{and} \quad \xi = \partial_s, \quad (2.1)$$

respectively. Moreover, $\Delta(\frac{1}{|\mathbf{x}|}) = 0$ for $\mathbf{x} \neq 0$ and therefore the metric is Ricci flat [5,6] – it is a vacuum solution of the Einstein equations. In other words, it is a *gravitational wave in 5D*.

The geodesics in Bargmann space are described by the Lagrangian

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$$\mathcal{L}_{geo} = \frac{1}{2}(\dot{\mathbf{x}})^2 + \dot{t}\dot{s} + \frac{GM_{\odot}}{|\mathbf{x}|} \dot{t}^2, \quad (2.2)$$

where the “dot”, $(\dot{}) = d/d\sigma$ is the derivative w.r.t. an affine parameter σ . Then (2.2) implies the equations of motion

$$\ddot{\mathbf{x}} = -GM_{\odot} \dot{t}^2 \frac{\mathbf{x}}{|\mathbf{x}|^3}, \quad (2.3a)$$

$$\ddot{t} = 0, \quad (2.3b)$$

$$\frac{d}{d\sigma} \left(\dot{s} + \frac{2GM_{\odot}}{|\mathbf{x}|} \dot{t} \right) = 0. \quad (2.3c)$$

The non-relativistic spacetime is identified with the quotient of \mathcal{M} by the integral curves of ξ ; the non-relativistic motions – in our case the Kepler orbits – are the projections to non-relativistic space-time of the null geodesic of the 5 dimensional metric (2.1).

Chrono-Projective transformations were introduced originally in the Newton-Cartan context [7]. In Bargmann terms they are conformal mapping of \mathcal{M} , $f^*g_{\mu\nu} = \Omega^2(t)g_{\mu\nu}$, which leave the direction of ξ invariant [8].¹ Working infinitesimally,

$$L_Y g_{\mu\nu} = 2\omega(t) g_{\mu\nu} \quad \text{and} \quad L_Y \xi = \psi(t) \xi. \quad (2.4)$$

Lifting the Kepler rescaling (1.1) to 5D Bargmann space as

$$t \rightarrow \Lambda^3 t, \quad \mathbf{x} \rightarrow \Lambda^2 \mathbf{x}, \quad s \rightarrow \Lambda s \quad (2.5a)$$

$$Y = 3t \partial_t + 2\mathbf{x} \partial_{\mathbf{x}} + s \partial_s \quad (2.5b)$$

rescales the metric conformally, $g_{\mu\nu} dx^\mu dx^\nu \rightarrow \Lambda^4 g_{\mu\nu} dx^\mu dx^\nu$. It does not preserve ξ , though, only its direction,

$$\partial_s \rightarrow \Lambda^{-1} \partial_s, \quad \text{i.e.} \quad L_Y \partial_s = -\partial_s, \quad (2.6)$$

and is therefore Chrono-Projective. The geodesic Lagrangian (2.2) scales, under the lifted Kepler scaling (2.5), as $\mathcal{L}_{geo} \rightarrow \Lambda^4 \mathcal{L}_{geo}$. At the first sight, this appears to be a no better behaviour than “downstairs”, (1.2), – and this is indeed so when the geodesic is timelike or spacelike. However for null geodesics the Lagrangian is constrained to vanish,

$$\mathcal{L}_{geo} = 0, \quad (2.7)$$

which makes it invariant: *lifted Kepler rescalings act, for null geodesics, as symmetries.*

Let us emphasise that “upstairs” i.e., in the 5D gravitational wave space-time, no additional constraint is required; Noether’s theorem works for any conformal vectorfield which leaves the geodesic Lagrangian invariant. The associated conserved quantity for motion along a null geodesic in 5D is, in our case,

$$\mathcal{Q} = 3tp_t + 2x^i p_i + sp_s, \quad (2.8)$$

whose conservation can also be checked by a direct calculation: in terms of the canonical momenta the equations of motion (2.3) imply that $p_t = \dot{s} + \frac{2GM_{\odot}}{|\mathbf{x}|} \dot{t}$ and $p_s = \dot{t}$ are constants of the motion. Then deriving \mathcal{Q} and using the equations of motion we get

$$\dot{\mathcal{Q}} = 4\mathcal{L}_{geo} = 0, \quad (2.9)$$

because, precisely, our geodesics are null. Conversely, the generating vector field Y in (2.5b) is recovered as $Y^\mu = \{x^\mu, \mathcal{Q}\} = \partial\mathcal{Q}/\partial p_\mu$.

¹ Transformations which project down are those which strictly preserve the vertical vector, $L_Y \xi = 0$ [4,5].

The geodesic Hamiltonian is

$$\mathcal{H}_{geo} = \frac{1}{2} p_i p_i - \frac{GM_{\odot}}{|\mathbf{x}|} p_s^2 + p_t p_s. \quad (2.10)$$

Performing a Legendre transformation, this Hamiltonian becomes the geodesic Lagrangian, $\mathcal{H}_{geo} = \mathcal{L}_{geo}$. Moreover, identifying p_s with m the mass in 3D and expressing p_t from the null condition $\mathcal{H}_{geo} = 0$ yields

$$p_t = - \left(\frac{1}{2m} p_i p_i - \frac{GmM_{\odot}}{|\mathbf{x}|} \right) = -E, \quad (2.11)$$

which is (minus) the non-relativistic energy. Then from $\dot{t} = p_s = m$ we infer that $d/d\sigma = md/dt$. Denoting d/dt by “prime”, $p_i = m(x^i)'$ and putting the geodesic Lagrangian equal to zero yields

$$s'(t) = - \left(\frac{1}{2} (x')^2 + \frac{GM_{\odot}}{|\mathbf{x}|} \right) = -\frac{1}{m} \mathcal{L}_{Kepler}. \quad (2.12)$$

The change of s along a lightlike geodesic is thus proportional to minus the 3D Kepler action calculated along the projected trajectory [6],

$$s(t) = \frac{\mathcal{Q}}{m} + 3t \frac{E}{m} - \frac{2(p_i x^i)}{m} = s_0 - \frac{1}{m} \int_0^t \mathcal{L}_{Kepler}(\mathbf{x}(\tau), \mathbf{x}'(\tau)) d\tau. \quad (2.13)$$

The conserved quantity (2.8) can be expressed “mostly” but *not completely* in 3D terms,

$$\mathcal{Q} = -3tE + m \frac{d}{dt} (x_i x^i) - \int_0^t \mathcal{L}_{Kepler}(\mathbf{x}(\tau), \mathbf{x}'(\tau)) d\tau + ms_0, \quad (2.14)$$

which explains also why it does not project down: it depends on s_0 . However subtracting ms_0 , we get $\mathcal{Q}_{Kepler} = \mathcal{Q} - ms_0$

$$\mathcal{Q}_{Kepler} = -3tE + m \frac{d}{dt} (x_i x^i) - \int_0^t \mathcal{L}_{Kepler}(\mathbf{x}(\tau), \mathbf{x}'(\tau)) d\tau \quad (2.15)$$

where the integration is along the classical trajectory in 3-space. \mathcal{Q}_{Kepler} is well-defined and also conserved, as proved along the same lines as for (2.9). We mention that the same expression can also be derived from the original Kepler Lagrangian using a generalization of the classical Noether theorem [9].

Let us stress that (2.15) is a *local quantity* despite its surprising form, because the classical trajectory (apart at caustic singularities) is uniquely determined by its end points. The integral is just Hamilton’s action function.

For $t = 0$ both the first and the last terms in (2.15) vanish, leaving us with $\mathcal{Q}_{Kepler} = 2p_i(0)x^i(0)$. Let us record for further reference that

$$\int_0^t \mathcal{L}_{Kepler} dt = \int_0^t p_i dx^i - Et, \quad \text{which allows us to rewrite (2.15) also as}$$

$$\mathcal{Q}_{Kepler} = -2tE + m \frac{d}{dt} (x_i x^i) - \int_0^t p_i \frac{dx^i}{dt} dt. \quad (2.16)$$

3. Applications of our conserved quantity

We restrict our attention henceforth to elliptic motions in the $x_3 = 0$ plane with $E < 0$ and draw some interesting consequences of the conservation of (2.15). Parabolic and hyperbolic motions behave similarly.

If time is measured so that for $t = 0$ we are in the closest (perihelion) position then Q_{Kepler} is just zero. But then a full period later i.e. for $t = T$, we are back where we started from, so that $(d(x_i x^i)/dt)(t = T) = 0$, yielding,

$$2TE + \int_0^T p_i \frac{dx^i}{dt} dt = 0. \quad (3.1)$$

The first consequence is that expressing the momenta in terms of velocities allows us to infer the *Virial Theorem: the energy is minus the average kinetic energy for a full period*,

$$E = -\frac{1}{T} \int_0^T m \left(\frac{d\mathbf{x}}{dt} \right)^2 dt = -\bar{E}_k. \quad (3.2)$$

The integral in (3.1) can actually be determined. Consider the radius vectors drawn from the focus where the Sun sits *and also from the other focus* to the current position of the planet. The rates of change of the areas swept out by these two radius vectors are

$$\frac{dA}{dt} = \frac{1}{2m} p_\phi, \quad (3.3a)$$

$$\frac{dA'}{dt} = \frac{b^2}{2p_\phi} \left(p_r \frac{dr}{dt} + p_\phi \frac{d\phi}{dt} \right), \quad (3.3b)$$

where p_ϕ is the angular momentum. The first of these relations is Kepler's Second Law, while (3.3b) might reasonably be called *Tait's Law* [10,11]; see [12] for a new, geometric proof. Then for a full period T both radius vectors sweep through the ellipse, and therefore,

$$\pi ab = \begin{cases} \int_0^T \frac{dA}{dt} dt = \frac{p_\phi}{2m} T, \\ \int_0^T \frac{dA'}{dt} dt = \frac{b^2}{2p_\phi} \int_0^T \left(p_r \frac{dr}{dt} + p_\phi \frac{d\phi}{dt} \right) dt, \end{cases} \quad (3.4)$$

where a and b are the semi-major and the semi-minor axes, respectively. From here we infer

$$\int_0^T \left(p_r \frac{dr}{dt} + p_\phi \frac{d\phi}{dt} \right) dt = \frac{4\pi^2 a^2 m}{T}. \quad (3.5)$$

Reinserting this into (3.1) and using $E = -GmM_\odot/2a$, we end up with the Third Law,

$$\frac{a^3}{T^2} = \frac{GM_\odot}{4\pi^2}. \quad (3.6)$$

Let us observe finally that while the planet goes around once the vertical coordinate changes, by (2.13), by

$$\begin{aligned} \Delta s &= -\frac{1}{m} \int_0^T \mathcal{L}_{Kepler} dt = -\frac{1}{m} \int_0^T \left(p_r \frac{dr}{dt} + p_\phi \frac{d\phi}{dt} \right) dt + \frac{E}{m} T \\ &= -\frac{4\pi^2 a^2}{T} - \frac{GM_\odot T}{2a}. \end{aligned} \quad (3.7)$$

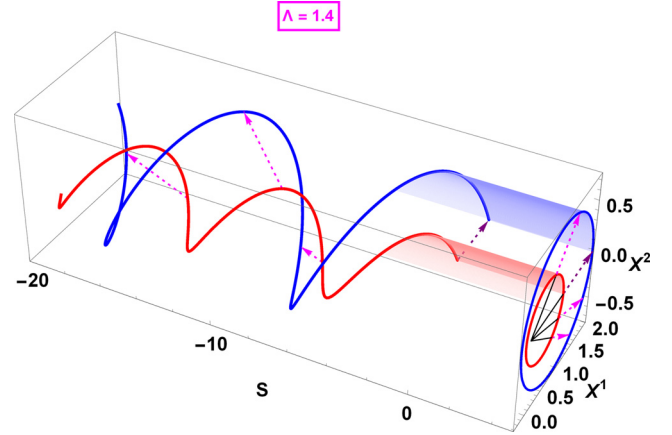


Fig. 1. The closed Keplerian trajectories become spirals when lifted to Bargmann space. They are permuted by the lift (2.5) of Kepler's rescaling (1.1) indicated by arrows.

The equations of motion (2.3) can be solved numerically; it confirms that (2.8) is indeed conserved, and also the formulae of this section. The solutions are shown in Fig. 1.

4. Generalization to N bodies

The Kepler's scaling property holds in fact for all of Newtonian Cosmology [12,13]. The N -body equations (no sum on a),

$$m_a \frac{d^2 \mathbf{x}_a}{dt^2} = - \sum_{b \neq a} m_a m_b \frac{\mathbf{x}_a - \mathbf{x}_b}{|\mathbf{x}_a - \mathbf{x}_b|^3}, \quad a = 1, 2, \dots, N \quad (4.1)$$

correspond, in the Bargmann framework, to the projections to the N -particle configuration space of the null geodesics of the $3N + 2$ dimensional metric [5],

$$g_{\mu\nu} dx^\mu dx^\nu = \sum_{a=1}^N \frac{m_a}{m} d\mathbf{x}_a^2 + 2dt ds - \frac{2U}{m} dt^2, \quad (4.2a)$$

$$m = \sum_a m_a, \quad U = -\frac{1}{2} \sum_{a,b \neq a} \frac{Gm_a m_b}{|\mathbf{x}_a - \mathbf{x}_b|}. \quad (4.2b)$$

Then the Kepler rescaling (2.5), $t \rightarrow \Lambda^3 t$, $\mathbf{x}_a \rightarrow \Lambda^2 \mathbf{x}_a$, $s \rightarrow \Lambda s$, acts plainly conformally, $g_{\mu\nu} dx^\mu dx^\nu \rightarrow \Lambda^4 g_{\mu\nu} dx^\mu dx^\nu$, generating a symmetry and a conserved charge for null geodesics,

$$Q = -3TE + 2 \sum_a \mathbf{x}_a \cdot \mathbf{p}_a + sm, \quad E = \sum_a \frac{\mathbf{p}_a^2}{2m_a} + U, \quad (4.3a)$$

$$s = s_0 - \frac{1}{m} \int_0^t \mathcal{L}_N d\tau, \quad \mathcal{L}_N = \sum_a \frac{m_a (\mathbf{x}'_a)^2}{2} - U. \quad (4.3b)$$

Here \mathcal{L}_N is the N -body Lagrangian. This charge projects again to a conserved charge of unconventional form.

5. Conclusion

This year we celebrate the 400'th anniversary of Kepler's discovery of his Third Law of planetary motion, which concerns the period and size of geometrically similar bound orbits [1].

Famously, Newton derived this and other properties of the orbits from his Universal Inverse Square Law of Gravitation. This is what we find in most textbooks, e.g. in [2]. Since his time there

have been many investigations of the geometry and symmetry of these orbits, but none has derived Kepler's Third Law using the methods introduced by Emmy Noether.

In this paper we start with a previously developed but not well known general formalism called the Bargmann framework of Eisenhart [3], and of Duval et al. [4,5]. It states that the motion may be regarded as the projection of the motion of light rays moving in a five-dimensional extended spacetime and obtain for the first time Kepler' law as a consequence of Emmy Noether's theorem.

In detail, lifting Kepler's rescaling, (1.1), to 5D Bargmann space as (2.5) generates there a well-behaved conserved charge, (2.8), for null geodesics. It allows us to integrate the "vertical" motion once the Kepler motions had been determined. Subtracting a constant term yields a conserved charge for ordinary planetary motion of an *unconventional form*.² Rather incredibly, (2.15) appears to be a new conserved quantity which seems to have escaped attention so far. It is in fact *different* from the familiar Runge-Lenz vector as can be understood by recalling their origin: while (2.8) is a scalar generated by a *conformal Killing vector* of 5D Bargmann space, the components of the Runge-Lenz vector are associated with 3 *Killing tensors* [5].

The conserved quantity (2.15) allows us to derive the Virial Theorem, (3.2), the usual form of Kepler's Third law, (3.6); the evolution of the s -coordinate is consistent with Fig. 1.

One can inquire if the Kepler problem admits further spacetime symmetries. The answer is *no*: the intrinsically defined Newton-Cartan structure allows for a 5-parameter Chrono-Projective group only, composed by rotations, time translations and the Kepler rescaling [7]. For further details and applications of conformal symmetries for gravitational waves, see [15,16]. Other examples of Chrono-Projective transformations include the Schrödinger-Newton equations [17], hydrodynamics [18], Schrödinger operators [19] and projective dynamics [20].

The expression $\int_0^T p_i \frac{dx^i}{dt} dt$ we used repeatedly in sec. 3 had ac-

tually played a prominent rôle in the *Old Quantum Theory*, namely in the Nicholson-Bohr-Wilson-Sommerfeld quantization conditions [21–25]: if a coordinate q varies periodically with time, then the quantity

$$\frac{1}{2\pi\hbar} \oint pdq, \quad (5.1)$$

where $\hbar = 2\pi\hbar$ is Planck's constant, should be an *integer*. For the closed Keplerian orbits we have two such coordinates, r and ϕ . We have in particular the quantization of angular momentum first suggested by Nicholson [21] and its generalization proposed, independently, by Bohr [22], Wilson [23,24] and by Sommerfeld [25],

$$\frac{1}{2\pi\hbar} \oint p_\phi d\phi = l, \quad \text{and} \quad \frac{1}{2\pi\hbar} \oint (p_r dr + p_\phi d\phi) = n, \quad (5.2)$$

where l the total angular momentum and n is the principal quantum number, respectively. The geometrical significance of these relations is given by (3.3b).

In our study we were helped by that, because of the *Equivalence Principle*, the Keplerian trajectories are independent of the mass. However, it is illuminating to consider the 3D transformations inherited from those in 5D phase space and generated by (2.8) by complementing (1.1) with

$$m \rightarrow \Lambda^{-1}m. \quad (5.3)$$

Details will be discussed elsewhere.

We conclude with the remark that Kepler's game changing three laws remain as relevant to our exploration of the universe and the laws that govern it today as when they were first formulated. No better illustration of this fact may be found than in [26, 27]. Studying the motion of matter around a black hole could provide a test for the validity/corrections of Kepler's laws at the large scale.

Note added

After our paper was submitted, we came across [28] which has a vague relation to our work here.

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² Unconventional conserved quantities for geodesic motion in a curved space were studied recently in [14].

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