

Conformal symmetries and integrals of the motion in pp waves with external electromagnetic fields

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Abstract

The integrals of the motion associated with conformal Killing vectors of a curved space-time with an additional electromagnetic background are studied for massive particles. They involve a new term which might be non-local. The difficulty disappears for pp-waves, for which explicit, local conserved charges are found. Alternatively, the mass can be taken into account by “distorting” the conformal Killing vectors. The relation of these non-point symmetries to the charges is analysed both in the Lagrangian and Hamiltonian approaches, as well as in the framework of Eisenhart-Duval lift.

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I. INTRODUCTION

Conserved quantities associated with the isometries play an important role for the integrability of the geodesic equations : the Killing vectors generate Noether symmetries and provide, consequently, integrals of the motion. (The argument can be extended to electromagnetic backgrounds, provided the latter are also preserved). Similar results hold for homothetic fields whose conformal factor is a constant [1–4], or for massless geodesics [5–7]. Recent applications involve the so called *Memory Effect* for gravitational waves, see [8–10] and references therein. However the procedure above breaks down for *proper conformal Killing vectors* whose conformal factor is not a constant and for massive geodesics.

Recently an alternative, non-Noetherian, approach to this problem has been proposed [11]. It has been shown that conformal Killing fields lead, except for some special parametrisation, to *non-local integrals of the motion* : the associated charge involves a novel type *integral term*,

$$m \int \omega_Y(x(\tau)) \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} d\tau. \quad (\text{I.1})$$

Calculating (I.1) requires to solve first the geodesic equations, making its use difficult.

The theory of ref. [11] is indeed the starting point for our considerations here. First we extend the result above to conformal fields Y ($\mathcal{L}_Y g = 2\omega_Y g$) which preserve also some electromagnetic background given by a potential A , i.e.,

$$\mathcal{L}_Y A = d\phi_Y, \quad (\mathcal{L}_Y F = 0) \quad (\text{I.2})$$

for some function ϕ_Y . For a geodesic of mass m the resulting charge is given in eqn. (II.3) below.

For a pp wave the charge can be calculated explicitly, raising several questions. Firstly, what is the relation (if any) between the conserved charges for pp-waves obtained above and the Noetherian or non-point-symmetry approach to the geodesic equations ? Secondly, can such charges provide new information or are they functions of known charges ? Thirdly, can we find electromagnetic backgrounds preserved by the conformal fields and the explicit forms of the corresponding integrals of the motion ?

Let us now recall that according to the Eisenhart-Duval approach [12–16] the two-dimensional classical dynamics can be lifted to massless geodesics, allowing us to recover the classical non-relativistic symmetries “downstairs” from those, relativistic, “upstairs”. Can

we describe the charges obtained above and find their meaning within this framework ?

This paper is devoted to answering these questions.

An important observation is that for massive geodesics proper conformal vector fields Y^α *can not be realised as a Noether point symmetries*. However we show that they can be generated instead by “distorting” Y^α in a mass-dependent way,

$$\Upsilon^\alpha = Y^\alpha + \frac{m^2}{p_v^2} f^\alpha, \quad (\text{I.3})$$

where p_v is the conserved “vertical” momentum and f^α is suitably defined vector field cf. (V.10) – so that the associated conserved charge reproduces the “non-local” one which will be constructed in sec.II, cf. (II.3). In particular, the integral term (I.1) is induced by “distorsion vector field” f^α .

These “distorted symmetries” are reminiscent of dynamical symmetries (whose typical example is the $\mathfrak{o}(4)$ symmetry of the Kepler problem involving the Laplace-Runge-Lenz vector) in that they are *not point symmetries* as (say) rotations or translations. Their generators involve also derivatives of the configuration space variables.

Our paper is organized as follows. In sec. II we generalize the results obtained in [1, 2, 11] to proper conformal fields and electromagnetic backgrounds preserved by them; we present both Lagrangian and Hamiltonian approaches. The explicit form of integrals of the motion for pp-waves is spelled out in sec. III. The relation to the Eisenhart-Duval lift [12–16] as well as further properties of the charges are obtained. Examples of electromagnetic backgrounds are presented in sec. IV. The relations between distorted symmetries and local charges is studied in sec. V. Further illustrations are contained in sec. VI.

II. “NON-LOCAL” INTEGRALS OF THE MOTION

A spinless particle in a relativistic space-time in the presence of additional vector potential A_μ is described by the Lagrangian

$$L = -m\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} + eA_\mu\dot{x}^\mu, \quad (\text{II.1})$$

which implies the equations of the motion

$$m(\ddot{x}^\mu + \Gamma_{\beta\alpha}^\mu \dot{x}^\beta \dot{x}^\alpha) = m\frac{d}{d\tau} \left(\ln \sqrt{-g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta} \right) \dot{x}^\mu + e\sqrt{-g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta} F^\mu{}_\nu \dot{x}^\nu, \quad (\text{II.2})$$

where the “dot” means derivation w.r.t. an arbitrary parameter τ .

In terms of constrained systems [17, 18] (II.1) is a singular Lagrangian and, due *parametrisation invariance*, the equations of the motion (II.2) are not all independent from each other : the action remains form-invariant under the infinite dimensional group of transformations $\tau \mapsto \tilde{\tau} = f(\tau)$ where f is an arbitrary function. Then, according to the *second Noether theorem* [18, 19], the equations of the motion satisfy an identity. Reparametrisation invariance implies that in a d -dimensional space-time only $d - 1$ of the x^μ are independent; and additional conditions can be imposed.

Let us now assume that Y is a conformal vector for the metric which leaves up-to-a-gauge transformation invariant also the vector potential, eqn. (I.2). Then a tedious calculation shows that for a conformal transformation Y with conformal factor $\omega \equiv \omega_Y$ and $\phi \equiv \phi_Y$, the quantity

$$I \equiv I_Y = \frac{m}{\sqrt{-g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}} Y_\mu \dot{x}^\mu + m \int \omega(x(\tau)) \sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} d\tau + e Y_\mu A^\mu - e\phi \quad (\text{II.3})$$

is a constant of the motion : $\dot{I} = 0$ along each trajectory, extending to electromagnetic fields the results presented in [11]. For an isometry ($\omega \equiv 0$), the non-local term vanishes and the well known result is recovered. Similarly, the (local) charge for a massless particle is obtained in the (singular) limit $m \rightarrow 0$.

The main disadvantage of such an integral of the motion is that it is in general *non local*: the explicit form of the trajectory is needed to calculate it. The charge may become local using a special parametrisation. We can, for example, perform the transformation [11] $s = s(\tau) = \int^\tau (-g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta)^{1/2} \omega(x(\tilde{\tau})) d\tilde{\tau}$ yielding a local expression,

$$\sqrt{-g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}} = \frac{1}{\omega(x(s))} \Rightarrow I = m(\omega Y_\mu \frac{dx^\mu}{ds} + s) + e Y_\mu A^\mu - e\phi; \quad (\text{II.4})$$

the price to pay in this generic time gauge is that the meaning of the geodesic equations becomes obscured.

A frequent choice is that of an *affine parameter* σ , characterized by the property

$$g_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} = -m^2. \quad (\text{II.5})$$

Then the equations of the motion (II.2) resp. the conserved quantity (II.3) become

$$\frac{d^2x^\mu}{d\sigma^2} + \Gamma_{\beta\alpha}^\mu \frac{dx^\beta}{d\sigma} \frac{dx^\alpha}{d\sigma} = eF_{\nu}^{\mu} \frac{dx^\nu}{d\sigma}, \quad (\text{II.6a})$$

$$I = Y_\mu \frac{dx^\mu}{d\sigma} + m^2 \int^\sigma \omega(x(\tilde{\sigma})) d\tilde{\sigma} + eY_\mu A^\mu - e\phi. \quad (\text{II.6b})$$

To conclude this section let us have a look from the Hamiltonian point of view at the integral of the motion in an arbitrary parametrisation. The Lagrangian (II.1) leads to an identically vanishing Hamiltonian. In order to generate dynamics in the phase space, we extend the configuration space by adding a new coordinate N (the Einbein [21]) and define an equivalent quadratic Lagrangian

$$\tilde{L} = \frac{1}{2N} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta - \frac{m^2}{2} N + eA_\alpha \dot{x}^\alpha, \quad (\text{II.7})$$

whose E-L equations reproduce (II.2). In terms of the Euler derivatives E_N and E_α for the degrees of freedom N and x^α , respectively, the Euler-Lagrange system consists of $d+1$ equations. One of them, namely $E_N(\tilde{L}) = \frac{\partial \tilde{L}}{\partial N} = 0$, is a constraint. Thus only $d-1$ of the remaining d equations $E_\alpha(\tilde{L}) = 0$, are independent : the constraint which involves only velocities sets a restriction among the d second order equations, reducing the number of truly independent relations.

The situation is analogous to what happens for Einstein's equations in four dimensions : you have 10 equations, 4 of which are constraints. As a consequence, of the 6 remaining equations only 6-4=2 are truly independent : General Relativity has two physical degrees of freedom.

Both cases are effected by Noether's second theorem and the existence of identities amongst the equations of motion.

The new variable N corresponds, through $E_N(\tilde{L}) = 0$, to

$$N(\tau) = m^{-1} \sqrt{-g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta}. \quad (\text{II.8})$$

Thus, putting $p_\alpha = N^{-1} g_{\alpha\beta} \dot{x}^\beta + eA_\alpha$,

$$\boxed{I = Y^\alpha p_\alpha + m^2 \int \omega(x(\tau)) N(\tau) d\tau - e\phi} \quad (\text{II.9})$$

is an integral of the motion, as confirmed in the Hamiltonian framework by the Dirac-Bergmann algorithm applied to \tilde{L} [17, 18]. The Hamiltonian and the two first class con-

straints are:

$$\mathcal{H} = \frac{N}{2} g^{\alpha\beta} (p_\alpha - eA_\alpha)(p_\beta - eA_\beta) + \frac{m^2}{2} N + p_N \dot{N}, \quad (\text{II.10a})$$

$$\phi_1 = p_N \approx 0, \quad \phi_2 = g^{\alpha\beta} (p_\alpha - eA_\alpha)(p_\beta - eA_\beta) + m^2 \approx 0. \quad (\text{II.10b})$$

Using the canonical equations of the motion for x^α and p_α and the condition (I.2), one obtains that

$$\dot{I} = \frac{\partial I}{\partial \tau} + \{I, \mathcal{H}\} = N\omega\phi_2 \approx 0. \quad (\text{II.11})$$

The total derivative of I is weakly zero in the Dirac sense [17, 18], i.e., vanishes whenever ϕ_2 does. Linear-in-the-momenta quantities with this property were called by Kuchař [20] conditional symmetries. Our (II.3) and (II.9), extend this notion to explicit dependence on the parameter τ .

To sum up, to any proper conformal vector which preserves also the electromagnetic background is associated a conserved charge. However for general parametrisation (including the affine one), the latter have a seemingly non-local contribution, which becomes local for trajectories with vanishing mass, $m = 0$.

III. CONFORMAL SYMMETRIES OF PP-WAVES

In this section we apply the procedure outlined above to pp-waves, and show that the charges related to conformal fields become local. Adopting the terminology of Ref. [22], we present the pp-wave metric in Brinkmann coordinates,

$$g = d\mathbf{x}^2 + 2dudv + H(u, \mathbf{x})du^2, \quad (\text{III.1})$$

where $\mathbf{x} = (x^1, x^2)$. Switching off the electromagnetic field, $A_\mu = 0$, we consider the extended Lagrangian \tilde{L} (II.7). The equations of the motion are

$$E_i \equiv \ddot{x}^i - \frac{1}{2} \partial_i H(u, x) \dot{u}^2 - \frac{\dot{N}}{N} \dot{x}^i = 0, \quad (\text{III.2a})$$

$$E_u \equiv \ddot{u} - \frac{\dot{N}}{N} \dot{u} = 0, \quad (\text{III.2b})$$

$$E_v \equiv \ddot{v} + \partial_i H(u, x) \dot{x}^i \dot{u} + \frac{1}{2} \partial_u H(u, x) \dot{u}^2 - \frac{\dot{N}}{N} \dot{v} = 0, \quad (\text{III.2c})$$

$$E_N \equiv H(u, x) \dot{u}^2 + 2\dot{u}\dot{v} + \delta_{ij} \dot{x}^i \dot{x}^j + N^2 m^2 = 0 \quad (\text{III.2d})$$

The u -equation (III.2b) is the simplest one as it does not contain the profile $H(u, x)$ explicitly. The canonical momenta read

$$p_i = \frac{1}{N} \delta_{ij} \dot{x}^j, \quad p_u = \frac{1}{N} (\dot{v} + H(u, x) \dot{u}), \quad p_v = \frac{\dot{u}}{N}, \quad (\text{III.3})$$

while p_N is weakly zero. The profile $H(u, x)$ does not depend on v , therefore its conjugate momentum p_v is a constant of the motion associated with the covariantly constant vector ∂_v . Note that for a massive particle the relation $p^\mu p_\mu = -m^2$ implies $p_v \neq 0$ as well as that

$$\dot{v} = -\frac{1}{2\dot{u}} \left((\dot{x}^i)^2 + H(u, x) \dot{u}^2 + m^2 N^2 \right), \quad (\text{III.4})$$

which is the algebraic solution of (III.2d) with respect to \dot{v} . Using the last equation in (III.3), the non-local part of the integral of the motion takes the form

$$\int \omega(x(\tau)) N(\tau) d\tau = \frac{1}{p_v} \int \omega(x(\tau)) \dot{u} d\tau. \quad (\text{III.5})$$

For pp-wave spacetimes this integral can be calculated explicitly, both for the type N or type O (conformally flat) cases. Let us recall some facts about conformal symmetries of a pp-wave space-time [23]. The components of a general conformal Killing vector are:

$$Y^u = \frac{\mu}{2} \delta_{ij} x^i x^j + a_i(u) x^i + a(u) \quad (\text{III.6a})$$

$$Y^v = -\mu v^2 + (x^i a'_i(u) + 2b(u) - a'(u)) v + M(u, x, y) \quad (\text{III.6b})$$

$$Y^i = -(\mu x^i + a_i) v + \gamma_{ijkl} a'_j(u) x^k x^l + b(u) x^i - \epsilon_{ij} c(u) x^j + c_i(u) \quad (\text{III.6c})$$

where μ is a non-zero constant and the ‘‘prime’’ $(\cdot)'$ refers to d/du . The corresponding conformal factor is

$$\omega = \omega(u, x^i, v) = b(u) + x^i a'_i(u) - \mu v. \quad (\text{III.7})$$

The function $M(u, x, y)$ in (III.6b) satisfies suitable consistency conditions [23] while for $H(u, x, y)$ the relation

$$[\mu x^i + a_i(u)] \partial_i H = 2\mu H + 2a''_i(u) x^i - 2a''(u) + 4b'(u) \quad (\text{III.8})$$

must hold with $\mu = \text{const.}$ ¹.

¹ δ_{ij} is the Kronecker delta, ϵ_{ij} the Levi-Civita symbol ($\epsilon_{12} = +1$) and $\gamma_{ijkl} = \frac{1}{2} \delta_{ij} \delta_{kl} - \epsilon_{ik} \epsilon_{jl}$.

Firstly, we observe that the integral (III.5) can be computed explicitly for conformal vector fields which are “chrono-projective” [7, 24], defined by the property

$$\mathcal{L}_Y \partial_v = \psi \partial_v \quad \text{where} \quad \psi = \text{const.} \quad (\text{III.9})$$

ψ here is the so-called chrono-projective constant. This condition is satisfied, e.g., by N-type null fluids and by exact plane gravitational waves [7].

Chrono-projectivity yields a relation between the conformal factor and the u -component of the vector field, $\omega \equiv \omega(u) = \frac{(Y^u)' - \psi}{2}$; see [7] for an explicit calculation.

Leaving exact gravitational waves to Section (VIC) we consider N-type null fluid pp-wave space-times [22]. The general form of the conformal Killing vector is

$$\omega(u) = \frac{1}{2}(a'(u) - \psi), \quad (\text{III.10a})$$

$$Y^u = a(u), \quad (\text{III.10b})$$

$$Y^i = \omega(u)x^i + c_i(u) + \gamma \epsilon_{ij}x^j, \quad (\text{III.10c})$$

$$Y^v = -\psi v + \frac{a''(u)}{4}\mathbf{x}^2 + c'_i x^i + E(u), \quad (\text{III.10d})$$

Since the conformal factor ω is a function of u only, the integrand in eq. (III.5) is a total derivative, allowing us to calculate the integral. We end up with a *local* conserved charge

$$I = Y^u \left(p_u + \frac{m^2}{2p_v} \right) + Y^v p_v + Y^i p_i - \frac{m^2}{2p_v} \psi u. \quad (\text{III.11})$$

$\psi = \text{const.}$ implies, in particular, that the last term is linear in u . Note also the shift $p_u \rightarrow p_u + \frac{m^2}{2p_v}$ in the momentum.

Using the definition of the canonical momenta (III.3), the quantity (III.11) can be expressed in terms of the velocities. Interestingly, *substitution of (III.4) cancels the mass terms* in (III.11). Thus the conserved charge for the massive geodesic (III.11) *coincides* with the one in the massless case. This statement holds for chrono-projective conformal fields.

The case of type O pp-waves is slightly more involved. However, when the conformal factor is a function of u only, the same arguments as for the type N apply, since (III.10a) still holds : the conformal factor is a total derivative with respect to u and the integral (III.5) can be carried out.

We are thus left to study fields as in (III.6) which lead to conformal factor $\omega = x^i a'_i(u) - \mu v$. For simplicity, we consider the affine parametrisation with $N = 1$ (II.5). The last eqn. of

(III.3) implies that u is proportional to the affine parameter (seen directly from (III.2b)), and (III.5) becomes $\frac{1}{p_v} \int \omega(x(u)) du$ in this case.

Let us stress that p_v is a constant of the motion and it can, if necessary, be rewritten explicitly in terms of the velocity $p_v = \frac{du}{d\sigma}$ where the parameter σ is defined by (II.5).

For any conformally flat pp-wave the profile H is (up to a coordinate transformation) a homogenous function of degree two in the transverse directions [23] and therefore the homothetic vector field

$$Y_h = x^i \partial_i + 2v \partial_v \quad (\text{III.12})$$

is conformal with $\omega = 1$. The corresponding integral of the motion is

$$I_h = p_v x^i x^{i'} + 2v p_v + \frac{m^2}{p_v} u. \quad (\text{III.13})$$

Since all pp-waves of the type O are homogenous of degree two, the following integral, along the trajectory, can be computed explicitly

$$\int v du = \int \left(\frac{1}{2} \mathbf{x} \cdot \mathbf{x}' - \frac{m^2 u}{2p_v^2} + \frac{I_h}{2p_v} \right) du = \frac{1}{4} \mathbf{x}^2 - \frac{m^2}{4p_v^2} u^2 + \frac{I_h}{2p_v} u. \quad (\text{III.14})$$

Now, by virtue of (III.7) the v variable enters the conformal factor at most linearly. Therefore, the integral in (III.5) can be computed, as can also the conserved charge.

There remains the case when the conformal factor for O type pp-waves is of the form $\omega = a_i' x_i$ (cf. (III.7)). Let us note that (III.8) implies that $\mathbf{a} = (a_1, a_2)$ obeys the same equation as \mathbf{x} does (see (III.2a) for $N = 1$). Along the trajectory we have therefore

$$\mathbf{a}' \cdot \mathbf{x} = \frac{1}{2} (\mathbf{a} \cdot \mathbf{x} + u \mathbf{x} \cdot \mathbf{a}' - u \mathbf{a} \cdot \mathbf{x}') \quad (\text{III.15})$$

and the integral of ω in (III.5) can again be explicitly computed.

In summary, for N-type pp-waves, the integrals of the motion can be computed in a straightforward manner ; for O-type pp-waves, the associated integrals of the motion can be computed explicitly in the affine parametrisation. This result can be extended to an arbitrary parametrisation, considered in sec. V.

Several questions can now be asked: do these conserved charges provide new information and can they be identified with Noether symmetries (as for Killing fields), or are they more general non-point symmetries ?

Next, can we find electromagnetic backgrounds preserved by the conformal fields and derive corresponding integrals of the motion? What is the meaning, in this context, of the Eisenhart-Duval lift ? We turn to these questions in the following sections.

IV. FURTHER ASPECTS OF INTEGRALS OF THE MOTION FOR PP-WAVES

A. Eisenhart-Duval lift and massive geodesics

According to the Eisenhart-Duval (E-D) framework [12–16], the non-relativistic dynamics in two space dimensions emerges from the light-like reduction of null geodesics in a $3 + 1$ dimensional relativistic space-time (III.1). Referring to the literature for details, we just sketch the main idea. In the u -parametrisation the geodesic equations (III.2) can be rewritten as,

$$\mathbf{x}'' = \frac{\nabla H}{2}, \quad (\text{IV.1a})$$

$$v' = -\frac{1}{2}(\mathbf{x}'^2 + H) - \frac{m^2}{2p_v^2}. \quad (\text{IV.1b})$$

The transversal part of the geodesic equations decouple and can be considered as the Newton equations with the u -coordinate viewed as non-relativistic time. The non-relativistic motions lifted to motion along geodesics with fixed m and p_v . (In the original approach, $m = 0$ and p_v is identified with the non-relativistic mass [12–14].)

For massless geodesics the integrals of the motion associated with conformal fields are local cf. (II.6b) and [6, 7]. Moreover, under some assumptions, these integrals project onto integrals of the motion for the underlying non-relativistic system.

In the previous section, we have noted that the conformal vector fields of N-type pp wave space-times satisfy the chrono-projectivity condition (III.9). Therefore, only their v -component depends on v according $Y^v = Y_1^v(u, \mathbf{x}) - \psi v$, see [7] and references therein.

The v -coordinate can be expressed in a non-local form in terms of the *action integral* of the projected non-relativistic dynamics,

$$v = - \int \frac{1}{2} \left(\mathbf{x}'^2 + H(u, x) + \frac{m^2}{p_v^2} \right) du = - \int L_{NR} du - \frac{m^2 u}{2p_v^2} + \text{const.}, \quad (\text{IV.2})$$

where L_{NR} is the non-relativistic Lagrangian. (For pp-waves of type O v can be obtained directly and consequently be eliminated).

The projected integral of the motion behaves similarly, as illustrated by the Kepler problem [4, 25]. It is worth to notice that the constraint $p_\mu p^\mu = -m^2$ in (II.10b) allows us to express p_u as

$$p_u = -\frac{p_i^2}{2p_v} + \frac{1}{2}H(u, x^i)p_v - \frac{m^2}{2p_v}, \quad (\text{IV.3})$$

which is (up to a constant) minus the non-relativistic Hamiltonian $H_{NR} = -p_u$ obtained by projection to transverse space.

The transverse eqs. (IV.1a) are identical for both the massive and massless geodesics; the only difference is that in the massive case the v coordinate is shifted by the linear-in- u term as in (IV.2). In the massless case $m = 0$ (IV.2) reduces to the horizontal lift [7, 12]. On the other hand we know how the v variable enters the conformal fields (III.6) and consequently the integrals of the motion associated with them (see the previous section). Thus we can find the difference between the massive and massless charges expressed in terms of the variables u, \mathbf{x} only.

Let us start with the conserved charges for a massive geodesic of a type N pp-wave. Using eq. (IV.2) v can be eliminated ; then the 4-dimensional integral of the motion (III.11) projects downstairs as,

$$I = p_v \left(-\frac{1}{2}(\mathbf{x}'^2 - H(u, \mathbf{x}))Y^u(u) + \mathbf{Y}(u, \mathbf{x}) \cdot \mathbf{x}' + Y_1^v(u, \mathbf{x}) + \psi \int L_{NR} du \right). \quad (\text{IV.4})$$

The terms which contain m reduce to a constant and we recover the charge for massless geodesics; expressed in terms of u, \mathbf{x} . However, for a proper chrono-projective transformation ($\psi \neq 0$) the integral term is non-local, unlike the expression (III.11) in full four-dimensional space-time.

The same situation holds for any conformal vector field of the pp-wave. Using the form of Y given in [22], (especially eqn. # (12)) one finds that after eliminating the v coordinate, the terms with mass reduce to a constant. Thus the conserved charge coincides with the massless one, expressed in terms of \mathbf{x} and u – just like the transverse parts of the geodesic equations for the pp-wave are identical for both massive and massless geodesics.

B. Conformally related metrics

Let us consider a pp-wave metric g and a new one \hat{g} conformally related to g ,

$$\hat{g} = \Omega^2(u) g, \quad (\text{IV.5})$$

where $\Omega(u)$ is an arbitrary function. Then \hat{g} has the same conformal algebra as g ; however, the Killing, homothetic and proper conformal transformations can be different ². On the

² These metrics are physically inequivalent. E.g., one can be a vacuum solution, the other not.

other hand, after a suitable transformation to new coordinates $\tilde{u}, \tilde{\mathbf{x}}, \tilde{v}$, the metric \hat{g} takes the pp-wave form [22]. As we have seen above, the integrals of the motion in a pp-wave for massive geodesics coincide with those of the massless ones when expressed in terms u, \mathbf{x} . However those integrals for g and \hat{g} , expressed in terms of u and \mathbf{x} , coincide up to a constant (this can be checked also directly using eqn. # (12) of [22]). Thus for both, physically inequivalent, pp-waves the integrals of the motion associated with the conformal generators can be directly related.

Let us illustrate this observation with the proper conformal transformations of a particular type of space-time which preserve, in addition, a given electromagnetic background.

In the Minkowski space-time there are four proper conformal transformations, see section (VI A). For example, we have the so-called standard special conformal vector (In the non-relativistic context, ∂_u is time translation and Y_K is an expansion [7]), see eqn. (VI.3b) below. In order to obtain geodesically complete metrics, we combine Y_K with the Killing vector ∂_u ,

$$Y^{(1)} = Y_K + \epsilon^2 \partial_u = (u^2 + \epsilon^2) \partial_u - \frac{1}{2} \mathbf{x}^2 \partial_v + u \mathbf{x} \cdot \nabla, \quad \omega(u) = u. \quad (\text{IV.6})$$

We focus our attention on gravitational or/and electromagnetic backgrounds. For example, in refs. [26–29] some electromagnetic as well as gravitational fields which satisfy (I.2) were studied. Due to the modification (IV.6) these backgrounds form non-singular pulses. Below we give further examples of gravitational fields for which $Y^{(1)}$ is a conformal vector, and more electromagnetic backgrounds which are preserved by this field.

Turning around the question, we ask which pp-wave space-times do admit $Y^{(1)}$ as conformal vector with identical conformal factor $\omega = u$. One finds that this happens iff its profile H satisfies

$$(u^2 + \epsilon^2) \partial_u H + u \mathbf{x} \cdot \nabla H + 2uH = 0. \quad (\text{IV.7})$$

The solution of this equation is of the form

$$H = \frac{2\epsilon^2}{u^2 + \epsilon^2} K\left(\frac{\mathbf{x}}{\sqrt{u^2 + \epsilon^2}}\right), \quad (\text{IV.8})$$

where K is an arbitrary function of two variables. Thus for this family of pp-waves the corresponding integral of the motion are (II.6b) (see also (IV.4)),

$$I^{(1)} = p_v \left(-\frac{1}{2} \mathbf{x}^2 + u \mathbf{x} \cdot \mathbf{x}' + \frac{1}{2} (-\mathbf{x}'^2 + H)(u^2 + \epsilon^2) - \frac{m^2}{2p_v^2} \epsilon^2 \right), \quad (\text{IV.9})$$

where the last term is just a mass-dependent constant.

To illustrate our observation let us now rewrite (IV.8) as

$$H = \frac{2\epsilon^2}{(u^2 + \epsilon^2)} \left(\tilde{K}\left(\frac{\mathbf{x}}{\sqrt{u^2 + \epsilon^2}}\right) + \frac{\mathbf{x}^2}{2(u^2 + \epsilon^2)} \right), \quad (\text{IV.10})$$

where \tilde{K} is arbitrary function. Then (IV.9) becomes

$$I^{(1)} = -\frac{\epsilon}{2}(\rho\mathbf{x}' - \rho'\mathbf{x})^2 + \epsilon^2\tilde{K}\left(\frac{\mathbf{x}}{\sqrt{\epsilon\rho}}\right) - \frac{\epsilon^2 m^2}{2p_v^2} \quad \text{where} \quad \rho = \sqrt{u^2 + \epsilon^2}/\sqrt{\epsilon}. \quad (\text{IV.11})$$

On the other hand, consistently with the general theory [30, 31] (and verified also by direct calculation), the Niederer transformation [32]

$$u = \epsilon \tan(\tilde{u}), \quad \mathbf{x} = \frac{\epsilon\tilde{\mathbf{x}}}{\cos(\tilde{u})}, \quad (\text{IV.12})$$

relates the transverse part of the geodesic equation to a “time”-independent set of equations,

$$\mathbf{x}'' = \tilde{\nabla}\tilde{K}(\tilde{\mathbf{x}}). \quad (\text{IV.13})$$

Next, let us note that the metric \tilde{g} defined by the profile $2\tilde{K}$ is conformally related to the metric g , defined by (IV.10). Indeed, supplying the transformation (IV.12) by $v = \epsilon\tilde{v} - \epsilon \tan(\tilde{u})\tilde{\mathbf{x}}^2/2$ one has

$$g = \frac{\epsilon^2}{\cos^2(\tilde{u})} \left(2\tilde{K}(\tilde{\mathbf{x}})d\tilde{u}^2 + 2d\tilde{u}d\tilde{v} + d\tilde{\mathbf{x}}^2 \right) = \frac{\epsilon^2}{\cos^2(\tilde{u})} \tilde{g}. \quad (\text{IV.14})$$

In the new variables the vector (IV.6) takes the form $\epsilon\partial_{\tilde{u}}$ and its conformal factor is $\epsilon \tan(\tilde{u})$. On the other hand, $\epsilon\partial_{\tilde{u}}$ is a Killing vector of \tilde{g} for which the suitable integral of the motion (II.6b) can easily be obtained; it turns out to be the energy for the projected dynamics,

$$\tilde{I} = \frac{1}{2}\tilde{\mathbf{x}}'^2 - \tilde{K}(\tilde{\mathbf{x}}) = E, \quad (\text{IV.15})$$

corresponding to eqs. (IV.13). Finally, the integral $I^{(1)}$ in the new variables takes the form

$$I^{(1)} = -\epsilon^2 \left(\tilde{I} + \frac{m^2}{2p_v^2} \right). \quad (\text{IV.16})$$

Thus, in full agreement with our general observations, in the massive case there is only a constant between projected and lifted integrals of the motion. The charges for both conformally related metrics g and \tilde{g} coincide.

C. Conformally invariant electromagnetic backgrounds

As mentioned already, if a Killing vector preserves the electromagnetic background, see (I.2), then one can construct a suitable integral of the motion (see e.g. [9] for a detailed discussions). The question is whether one can find a pp-wave admitting a conformal field which preserves some electromagnetic backgrounds and the corresponding charge localizes in affine parametrisation. We give here some examples of such a situation (extending some electromagnetic vortices, see [29] for further discussion).

Let us consider the pp-wave space-time defined by the profile (IV.8) (in particular, the Minkowski space-time). Then the field $Y^{(1)}$, eqn. (IV.6), is a conformal one. Now we take an electromagnetic field, $\mathcal{A} = A_u du = A(u, \mathbf{x}) du$, where A is an arbitrary function. Then p_v is again a constant of the motion and u is proportional to the affine parameter. Let us impose the condition (I.2), i.e. we assume that the potential is preserved by $Y^{(1)}$ up to a gauge transformation ϕ . Straightforward computations imply that ϕ is a function of u only; however, then one can find a suitable gauge transformation of the electromagnetic potential such that $\phi = 0$; thus, without loss of generality, we can assume this condition (for a fixed field Y such a choice of ϕ is always possible; however, not necessary explicitly given). Then eqn. (I.2) imposes only one condition on the profile A , which is, remarkably, of the same form as (IV.7) (after the substitution $H \rightarrow A$). Thus we obtain a whole family (cf. eqn. (IV.8)) of electromagnetic profiles which are preserved by $Y^{(1)}$. Of course, one can put $A = H$ (cf. the double copy conjecture [33]); however, A and H can be chosen independently (for example, we can take $H = 0$ i.e. Minkowski space-time). For such pp-waves and electromagnetic fields the integral of the motion associated with $Y^{(1)}$ can be written down explicitly, see eqn. (II.6b),

$$I_A^{(1)} = I^{(1)} + \frac{e}{p_v}(u^2 + \epsilon^2)A, \quad (\text{IV.17})$$

where $I^{(1)}$ is given by (IV.9).

Let us note finally that such an integral of the motion can bring some new information even for the Minkowski space-time. Indeed, the electromagnetic fields constructed are preserved by ∂_v which leads to charge p_v ; however, symmetries related to other Killing fields (Poincaré generators) are in general broken and do not provide integrals of the motion (see also the discussion in sec. V C as well as in [29]).

V. DISTORTED CONFORMAL SYMMETRIES

In this section we first review some general aspects of the Noether symmetry approach [38–40] and explain how the previously found integrals of the motion are related to symmetry transformations of the action. Our main statement is that modifying the standard procedure allows us to derive the generators of the conserved charges by a mass -dependent “distortion” of the conformal Killing vectors. The connection of these charges to a more general symmetry is also established and their general properties in phase space are investigated.

A. The Noether symmetry approach to geodesic systems

Let us first recall some basic facts [38–40]. A transformation is a Noether symmetry if it leaves the action integral form-invariant up to a surface term. We use the Lagrangian (II.7) as a model to illustrate the basic properties of the general theory. In infinitesimal form the symmetry generator is

$$X = \chi \frac{\partial}{\partial \tau} + \Upsilon^N \frac{\partial}{\partial N} + \Upsilon^\alpha \frac{\partial}{\partial x^\alpha}. \quad (\text{V.1})$$

Its extension to the space of the first derivatives ,

$$\text{pr}^{(1)}X = X + \left(\frac{d\Upsilon^\alpha}{d\tau} - \dot{x}^\alpha \frac{d\chi}{d\tau} \right) \frac{\partial}{\partial \dot{x}^\alpha} \quad (\text{V.2})$$

(called the first prolongation of X) is required to satisfy the infinitesimal invariance criterion,

$$\boxed{\text{pr}^{(1)}X(L) + L \frac{d\chi}{d\tau} = \frac{dF}{d\tau}} \quad (\text{V.3})$$

for some function F .

Equation (V.3) is our starting point for searching for symmetries. For a given Lagrangian L one tries to find appropriate vectors X and corresponding functions F which satisfy (V.3). The conserved quantity associated with a symmetry generator X is,

$$\boxed{Q = \Upsilon^\alpha \frac{\partial L}{\partial \dot{x}^\alpha} + \chi \left(L - \dot{x}^\alpha \frac{\partial L}{\partial \dot{x}^\alpha} \right) - F.} \quad (\text{V.4})$$

(A term $\Upsilon^N \frac{\partial L}{\partial N}$ could also be included in (V.2), but for the Lagrangian \tilde{L} of (II.7) such additions are trivial and are therefore omitted.)

$L = L_0 := \frac{1}{2}g_{\alpha\beta}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}$			$L = \tilde{L} := \frac{1}{2N}g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta - N\frac{m^2}{2}$				Geometric conditions
$\chi(\sigma, x)$	$\Upsilon(\sigma, x)$	$F(\sigma, x)$	$\chi(\tau, N, x)$	$\Upsilon^N(\tau, N, x)$	$\Upsilon(\tau, N, x)$	$F(\tau, N, x)$	on space-time vectors Y_0, Y_h
const.	-	const.	$\chi(\tau)$	$\dot{\chi}(\tau)N$	-	const.	-
-	$Y_0(x)$	const.	-	-	$Y_0(x)$	const.	$\mathcal{L}_{Y_0}g_{\alpha\beta} = 0$
-	$\sigma Y_0(x)$	$\Phi(x)$	-	-	-	-	if also $Y_0^\alpha = \nabla^\alpha\Phi$
$2h\sigma$	$Y_h(x)$	const.	-	-	-	-	$\mathcal{L}_{Y_h}g_{\alpha\beta} = 2hg_{\alpha\beta}, h = \text{const.}$
$h\sigma^2$	$2\sigma Y_h(x)$	$\Sigma(x)$	-	-	-	-	if also $Y_h^\alpha = \nabla^\alpha\Sigma$

TABLE I: All point symmetry solutions of (V.3) for affinely parametrised and resp. parametrisation-invariant Lagrangians.

The simplest case is to consider *Noether point symmetries*, where the generators do not depend on the derivatives. That is, we can have at most :

$$\chi = \chi(\tau, N, x), \quad \Upsilon^N = \Upsilon^N(\tau, N, x), \quad \Upsilon = \Upsilon^\alpha(\tau, N, x). \quad (\text{V.5})$$

Such a case can be treated algorithmically since the coefficients of the derivative terms (\dot{N}, \dot{x} and their powers) appearing in (V.3) must be zero. In this manner a set of over-determined linear, partial differential equations is obtained. Their solution (if it is not trivial, i.e. $X = 0, F = \text{const.}$) yields a Noether point symmetry.

On the other hand, if the coefficients in (V.1) are allowed to depend also on derivatives of the configuration space variables, e.g. $\chi = \chi(\tau, N, x, \dot{N}, \dot{x}, \dots)$ etc., then the previous procedure cannot be followed and (V.3) has to be treated in its totality as a single master equation. For this reason, non-point (or generalized) symmetries are much more difficult to be extracted.

For the geodesic problem, the list of Noether point symmetries is well known [35–37, 41, 43]. In the massive case the basic Noether symmetry generators are related to the homothetic algebra of the metric [35, 36, 43]. For null geodesics however, the results can be extended to include all conformal Killing fields, Y , [43, 44]. (See also [7] for a different approach). In Table I we collect the known results on point symmetries for the geodesic action both for the affinely parametrised case $L_0 = \frac{1}{2}g_{\alpha\beta}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}$ and for the parametrisation-invariant Lagrangian \tilde{L} [34]. In the former case there is no N field, while in the latter the Einbein field is considered as a degree of freedom on equal footing with x^α .

In the affine case – apart from the trivial time translation which implies that the Hamil-

tonian is a constant – the Killing and homothetic fields, Y_0 and Y_h respectively, contribute Noether point symmetries. What is more, both can be used to provide an additional conserved charge when they happen to be gradient vectors [35]. On the contrary, for the parametrisation invariant Lagrangian the generator for arbitrary transformations in time ($\chi(\tau)$ remains an arbitrary function) and the integrals of the motion generated by Killing vectors Y_0 are obtained [34].

Let us turn to the pp-wave space-time (III.1) and consider a free particle of mass m described by the Euler-Lagrange equations (III.2) (with $A_\alpha = 0$). As we noted in sec. III the localization of charges is related to the fact that on shell the conformal factor ω is the *total time derivative of a suitable function*,

$$\omega(x^\alpha(u)) = (f^\alpha x'_\alpha)' \quad (\text{V.6})$$

where

$$f^u = 0, \quad (\text{V.7a})$$

$$f^v = \frac{1}{2}u (x^i a'_i(u) - a'(u) + 2b(u) - 2\mu v) + \frac{1}{2}x^i a_i(u) + \frac{\mu}{4}\delta_{ij}x^i x^j + \frac{1}{2}a(u) - \frac{m^2}{p_v^2} \frac{\mu}{4}u^2, \quad (\text{V.7b})$$

$$f^i = -\frac{1}{2}u (\mu x^i + a_i(u)) . \quad (\text{V.7c})$$

Then along trajectories we have see (III.2) and (III.3)

$$f^\alpha p_\alpha = \frac{f^\alpha \dot{x}_\alpha}{N} = f^\alpha x'_\alpha \frac{\dot{u}}{N} \Rightarrow \frac{d}{d\tau}(f^\alpha p_\alpha) = \frac{\dot{u}}{N}(f^\alpha x'_\alpha)' \dot{u} = (f^\alpha x'_\alpha)' p_v^2 N = \omega p_v^2 N.$$

Thus we obtain

$$m^2 \int \omega(x(\tau)) N(\tau) d\tau = \frac{m^2}{p_v^2} f^\alpha p_\alpha + \text{const.} = \frac{m^2}{N p_v^2} f^\alpha \dot{x}_\alpha + \text{const.} \quad (\text{V.8})$$

The *charge is local*.

Note that (V.8) is valid in any arbitrary parametrisation (because the momenta are parametrisation invariant). Therefore the integral of the motion I given by (II.9) is equivalent to

$$Q = \Upsilon^\alpha \frac{\partial \tilde{L}}{\partial \dot{x}^\alpha}, \quad (\text{V.9})$$

associated with the modified (“distorted”) vector field

$$\boxed{\Upsilon^\alpha = Y^\alpha + \frac{m^2}{p_v^2} f^\alpha.} \quad (\text{V.10})$$

We now illustrate the new formulation by examples:

- Homothety : the comparison of (III.12) with eqn. (III.6) yields

$$Y^u = 0, \quad \Rightarrow \quad \mu = a_i = a(u) = 0 \quad (\text{V.11a})$$

$$Y^v = 2v \quad \Rightarrow \quad b = 1, \quad M = 0 \quad (\text{V.11b})$$

$$Y^i = x^i \quad \Rightarrow \quad \gamma_{ijkl} = \gamma(u) = c_i(u) = 0, \quad (\text{V.11c})$$

with $\omega = b = 1$. Eqns. (V.7) give $f^u = f^i = 0$, $f^v = u$. Then (V.10) used in (V.9) implies the integral of motion (III.13), as expected.

- pp-waves of the type N: from (III.10) we get

$$Y^u = a(u) \quad \Rightarrow \quad \mu = a_i = 0, \quad (\text{V.12a})$$

$$\omega(u) = \frac{a'(u) - \psi}{2} \quad \Rightarrow \quad b(u) = \frac{a'(u) - \psi}{2}, \quad (\text{V.12b})$$

$$Y^v = -\psi v + \frac{a''(u)}{4} \mathbf{x}^2 + c'_i x^i + E(u) \Rightarrow M = \frac{a''(u)}{4} \mathbf{x}^2 + c'_i x^i + E(u) \quad (\text{V.12c})$$

$$Y^i = \omega(u) x^i + c_i(u) + \gamma \epsilon_{ij} x^j \Rightarrow \text{automatically satisfied.} \quad (\text{V.12d})$$

Thus (V.7) is solved by $f^u = f^i = 0$, $f^v = \frac{1}{2}(Y^u - u\psi)$, which results through (V.9) to a conserved charge consistent with (III.11).

By comparing (V.4) with $\chi = F = 0$ and $L = \tilde{L}$ derived from the Noether symmetry approach with the conserved quantity in (V.9) obtained with no reference to Noether symmetry suggests that the latter is generated by the modified vectorfield Υ in (V.10), which plays a role similar to a Noether point-symmetry. We will return to this point later. However it can be easily checked that (V.10) fails to satisfy the symmetry criterion (V.3). It is only in the $m = 0$ case that we recover what we know from the general theory, namely that for null geodesics the conformal Killing vectors Y yield linear-in-the momenta conserved quantities [43, 44].

An intriguing observation is that the vectors Υ of (V.10) do not necessarily close to an algebra. They trivially do so when $m = 0$, where they reduce to the conformal Killing vectors of the metric. This leads us to inquiring whether the Noether symmetry approach can be modified so that it explains the form of the “distorted” vector (V.10) when $m \neq 0$ and the emergence of the conserved charge (V.9).

B. Modification of the Noether approach and the role of the constraints

In the previous subsection we outlined the procedure of deriving a point symmetry generator satisfying (V.3). Take for example the quadratic parametrisation-invariant Lagrangian \tilde{L} of (II.7) for space-time (III.1). The invariance criterion (V.3) requires to solve the system of partial differential equations for χ , Υ^N and Υ which demands the coefficients of \dot{u} , \dot{v} , \dot{x} , \dot{y} to vanish. This scenario leads directly to the right part of Table I, which, – leaving out the parametrisation invariance – is equivalent to the Killing equations. Instead of following strictly this procedure, we choose to modify it in a manner that is consistent with the equations of motion.

First we notice that it is possible to express the velocity \dot{v} in terms of the remaining variables, see eqn. (III.4). Then we eliminate N by using $N = \dot{u}/p_v$ which is the first integral of (III.2b), see the last of eq. (III.3). Substituting into (V.3) and collecting the coefficients of the remaining velocities \dot{x} and \dot{u} leads us to the following weaker conditions on Υ . (For the rest we consider $\chi = \Upsilon^N = 0$, $F = \text{const.}$).

$$\frac{\partial \Upsilon^u}{\partial v} = 0, \quad \frac{\partial \Upsilon^1}{\partial y} + \frac{\partial \Upsilon^2}{\partial x} = 0, \quad \frac{\partial \Upsilon^u}{\partial x} + \frac{\partial \Upsilon^1}{\partial v} = 0, \quad \frac{\partial \Upsilon^u}{\partial y} + \frac{\partial \Upsilon^2}{\partial v} = 0, \quad (\text{V.13a})$$

$$\frac{\partial \Upsilon^u}{\partial u} - \frac{m^2}{p_v^2} \frac{\partial \Upsilon^u}{\partial v} + \frac{\partial \Upsilon^v}{\partial v} - 2 \frac{\partial \Upsilon^2}{\partial y} = 0, \quad (\text{V.13b})$$

$$\frac{\partial \Upsilon^u}{\partial u} - \frac{m^2}{p_v^2} \frac{\partial \Upsilon^u}{\partial v} + \frac{\partial \Upsilon^v}{\partial v} - 2 \frac{\partial \Upsilon^1}{\partial x} = 0, \quad (\text{V.13c})$$

$$\left(H - \frac{m^2}{p_v^2}\right) \frac{\partial \Upsilon^u}{\partial y} + 2 \frac{\partial \Upsilon^v}{\partial y} + 2 \frac{\partial \Upsilon^2}{\partial u} - \left(H + \frac{m^2}{p_v^2}\right) \frac{\partial \Upsilon^2}{\partial v} = 0, \quad (\text{V.13d})$$

$$\left(H - \frac{m^2}{p_v^2}\right) \frac{\partial \Upsilon^u}{\partial x} + 2 \frac{\partial \Upsilon^v}{\partial x} + 2 \frac{\partial \Upsilon^1}{\partial u} - \left(H + \frac{m^2}{p_v^2}\right) \frac{\partial \Upsilon^1}{\partial v} = 0, \quad (\text{V.13e})$$

$$\begin{aligned} &\left(H - \frac{m^2}{p_v^2}\right) \frac{\partial \Upsilon^u}{\partial u} + \frac{\frac{m^4}{p_v^4} - H^2}{2} \frac{\partial \Upsilon^u}{\partial v} + 2 \frac{\partial \Upsilon^v}{\partial u} - \left(H + \frac{m^2}{p_v^2}\right) \frac{\partial \Upsilon^v}{\partial v} + \Upsilon^u \frac{\partial H}{\partial u} \\ &+ \Upsilon^1 \frac{\partial H}{\partial x} + \Upsilon^2 \frac{\partial H}{\partial y} = 0, \end{aligned} \quad (\text{V.13f})$$

These equations differ from those satisfied by a conformal Killing vector only in terms which involve the mass, m . The “distorted” Υ in (V.10) satisfies the above set of equations; consequently, the modification of the Noether procedure by invoking the known integrals of the motion (III.3), (III.4) before collecting coefficients, leads us to the desired fields.

As far as the conservation of Q in (V.9) is concerned, it is straightforward to show that

$$\frac{dQ}{d\tau} = -2N\omega_m E_N(\tilde{L}) - \Upsilon^\alpha(u, v, x, y) E_\alpha(\tilde{L}) + \frac{m^2}{N} \omega_m \left(N^2 - \frac{\dot{u}^2}{p_v^2} \right), \quad (\text{V.14})$$

where $E_N(\tilde{L}) = 0$, $E_\mu(\tilde{L}) = 0$ are the Euler-Lagrange equations of (III.2) and

$$\omega_m = \omega - \frac{m^2}{2p_v^2} \mu u. \quad (\text{V.15})$$

From (V.14) we see that the right hand side provides us with an additional condition which is satisfied on mass shell: the first integral of (III.2b), $\dot{u} = p_v N$, with p_v a constant. The Υ defined by (V.10) is a vector field on the configuration space only provided p_v entering its right hand side is viewed as a parameter. It does not formally define a point symmetry; this is clearly seen from eqn. (V.14) because the right hand side is not just a combination of Euler-Lagrange equations. However, if we restrict ourselves to trajectories whose momentum conjugated to v takes the value p_v (entering (V.10)), then the last term on the right hand side vanishes and we obtain the desired conservation law.

Now an additional question arises: Can the Υ be related to some formal Noether symmetry that directly satisfies (V.3)? The answer is affirmative: we just need to eliminate the constants of the motion in the vector (V.10) by their velocity-equivalents on mass shell. With this substitution in (V.10) the new generalized vector Υ satisfies the symmetry criterion (V.3) on the mass shell ($\chi = \Upsilon^N = 0$, $F = \text{const.}$) without the need to involve in addition the constraint equation or an integral of the motion. What is more, this symmetry (which is now a non-point Noether symmetry due to its dependence on velocities) *can* satisfy (V.3), not just for $L = \tilde{L}$, but also for the affinely parametrised Lagrangian L_0 .

The above procedure can be realized also at the Hamiltonian level. Take the Killing tensor denoted by $K_{\mu\nu} = (\partial_v \otimes \partial_v)_{\mu\nu} = \delta_{\mu\nu} \delta_{\nu v}$. Then the system possesses the conserved charge $\mathcal{K} = K^{\alpha\beta} p_\alpha p_\beta$. Let us promote the corresponding conservation law to a constraint,

$$\phi_3 \equiv K^{\alpha\beta} p_\alpha p_\beta - \kappa \approx 0, \quad (\text{V.16})$$

where κ is the constant value taken by \mathcal{K} along a trajectory ($\kappa = p_v^2$ is used here so as to avoid confusion with seeing p_v as a variable on the phase space). Integrals of the motion viewed as first class constraints have previously been studied in [42] from the perspective of the gauge transformations they generate.

The constraint (V.16) is consistent with the evolution of the system since it does not generate additional restrictions : $\dot{\phi}_3 = \{\phi_3, \mathcal{H}\} = 0$. It also commutes with both $\phi_1 = p_N$ and $\phi_2 = g^{\mu\nu}p_\mu p_\nu + m^2$ which makes it a first class constraint. Considering a quantity which is linear in the momenta, $Q = \Upsilon^\alpha p_\alpha$, and imposing the condition $\dot{Q} \approx 0$; a conditional symmetry in Kuchař's sense emerges. Assuming the additional constraint ϕ_3 this condition can be rewritten as,

$$\begin{aligned} \dot{Q} = \{Q, \mathcal{H}\} \approx 0 &\Rightarrow \dot{Q} = \omega_m(x)N\phi_2 + \tilde{\omega}(x)N\phi_3 \\ &= \omega_m N (g^{\mu\nu}p_\mu p_\nu + m^2) + \tilde{\omega} N K^{\mu\nu} p_\mu p_\nu - \omega_2 N \kappa, \end{aligned} \quad (\text{V.17})$$

where the multiplying factors on the right hand side are chosen to be consistent with what appears on the left. The left hand side is purely quadratic in the momenta, hence we demand $\tilde{\omega} = \frac{m^2}{\kappa}\omega_m$, which yields

$$\{Q, \mathcal{H}\} = \omega_m N \left(g^{\mu\nu} + \frac{m^2}{\kappa} K^{\mu\nu} \right) p_\mu p_\nu, \quad (\text{V.18})$$

cf. (II.11), leading subsequently to the geometric condition

$$\mathcal{L}_\Upsilon g_{\mu\nu} = 2\omega_m \left(g_{\mu\nu} + \frac{m^2}{\kappa} K_{\mu\nu} \right). \quad (\text{V.19})$$

This is the relation satisfied by the distorted field Υ in (V.10), where both m^2 and $p_v^2 = \kappa$ are to be understood strictly as constants, with $\omega_m = \omega - \frac{m^2}{2\kappa}\mu u$. In consequence, the above-modified Noether procedure has a Hamiltonian counterpart and the constraint ϕ_3 is necessary for its realization. Note that (V.19) can be reproduced for a generic metric with a (reducible or irreducible) Killing tensor $K_{\mu\nu}$, which means that this type of extended family of conserved charges may emerge in other cases, not just for pp-wave geodesics.

The intermediate situation where the p_v^2 in Q is not considered as a constant κ but as dynamical, needs only the constraint $\phi_2 \approx 0$ to be satisfied, i.e. $\{Q, \mathcal{H}\} \propto \phi_2 \approx 0$. On the contrary, the integral of the motion obtained by additionally substituting $m^2 = -g^{\alpha\beta}p_\alpha p_\beta$ commutes directly with the Hamiltonian without the need of any constraint. The generic properties of this conserved quantity are studied explicitly in the next section.

To sum up, we have demonstrated the existence of a higher order (non-point) symmetry that yields an integral of the motion which is rational in the momenta. The subsequent use of the constraints $\phi_2 \approx 0$ and $\phi_3 \approx 0$ is what helps us reduce the latter to an (on mass shell) equivalent linear expression generated by the distorted conformal vector (V.10) that we obtained through our modification. In Table II we collect the resulting expressions.

Generator	Conserved charge in phase space	Necessary conditions
$Y + \frac{m^2}{\kappa} f$	$Y^\alpha p_\alpha + \frac{m^2}{\kappa} f^\alpha p_\alpha$	$\phi_2 \approx 0, \phi_3 \approx 0$
$Y + \frac{N^2 m^2}{\dot{u}^2} f$	$Y^\alpha p_\alpha + m^2 \frac{f^\alpha p_\alpha}{p_v^2}$	$\phi_2 \approx 0$
$Y - \frac{H\dot{u}^2 + 2\dot{u}\dot{v} + \delta_{ij}\dot{x}^i\dot{x}^j}{\dot{u}^2} f$	$Y^\alpha p_\alpha - \frac{g^{\mu\nu} f^\alpha p_\alpha p_\mu p_\nu}{p_v^2}$	-

TABLE II: The integrals of the motion involving conformal Killing vectors Y for pp-wave geodesics and the conditions needed to commute with \mathcal{H} . The constraints allow for a complicated rational integral of the motion (bottom line) to be expressed in linear form (first line).

Finally, let us briefly consider the mass-distorted metric

$$g_{\alpha\beta}^{(m)} dx^\alpha dx^\beta = dx^2 + dy^2 + 2dudv + (H(u, x, y) + m^2)du^2, \quad (\text{V.20})$$

emanating from the right hand side of (V.19). A remarkable observation is that the mass- m geodesics of the metric g in (III.1) are in fact *massless geodesics of the mass-distorted metric* (V.20). To see this we fix m and consider the extended Lagrangian (II.7) for a geodesic with mass parameter M_0 (to be fixed later) in the deformed metric (V.20)

$$\tilde{L}_m = \frac{1}{2N} g_{\alpha\beta}^{(m)} \dot{x}^\alpha \dot{x}^\beta - \frac{M_0^2}{2} N. \quad (\text{V.21})$$

By inspection, the equations (III.2a)–(III.2b) for the metric g are identical to the corresponding Euler-Lagrange equations of \tilde{L}_m . But we also have the constraint equations

$$(H(u, x) + m^2) \dot{u}^2 + 2\dot{u}\dot{v} + \delta_{ij}\dot{x}^i\dot{x}^j + N^2 M_0^2 = 0 \quad \text{for } g^{(m)} \text{ with mass } M_0 \quad (\text{V.22a})$$

$$H(u, x)\dot{u}^2 + 2\dot{u}\dot{v} + \delta_{ij}\dot{x}^i\dot{x}^j + N^2 m^2 = 0 \quad \text{for } g \text{ with mass } m. \quad (\text{V.22b})$$

The Euler-Lagrange equation for v implies $N = \frac{\dot{u}}{p_v}$ in both cases. Substituting this into (V.22b) it becomes identical to (V.22a) if $M_0 = 0$, proving our statement.

C. Conserved charges and canonical symmetries

In this section we discuss the relation between conformal transformations and conservation laws in the Hamiltonian framework, based on the affine parametrisation. Such an approach has some advantages: we are dealing with the unconstrained Hamiltonian formalism thus we do not need to use the notion of conditional symmetry. (Ordinary canonical

transformations are admitted). Moreover, returning to the Lagrangian formalism is straightforward.

We start with the Hamiltonian $\mathcal{H}_a = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu$ on the phase space (x, p) equipped with the Poisson bracket $\{x^\mu, p_\nu\} = \delta_\nu^\mu$. Then given an arbitrary conformal field Y one finds that G_Y , defined as

$$G_Y = G_Y(x, p) = Y^\mu(x)p_\mu, \quad (\text{V.23})$$

obeys

$$\{G_Y, G_{Y'}\} = G_{[Y', Y]}, \quad \{G_Y, \mathcal{H}_a\} = 2\omega_Y \mathcal{H}_a, \quad (\text{V.24})$$

and G_Y generate canonical transformations on the phase space. The algebra generated by G_Y 's is (anti) isomorphic to the relevant conformal algebra. However, G_Y are conserved if either Y is a genuine Killing field, $\omega_Y = 0$, or if we are considering trajectories which lie on the invariant submanifold $\mathcal{H}_a = 0$. In the general case the second equation (V.24) allows us to construct the new conserved quantity

$$\tilde{G}_Y = G_Y - 2\mathcal{H}_a \int^\sigma \omega_Y(\tilde{\sigma}) d\tilde{\sigma}. \quad (\text{V.25})$$

\tilde{G}_Y is a non-local expression in general. However in some particular cases, as, for example, for pp-metrics it can be local. In such cases we obtain

$$\tilde{G}_Y = G_Y - \Omega_Y \mathcal{H}_a + \text{irrelevant terms}, \quad (\text{V.26})$$

where $\Omega_Y = \Omega_Y(x, p)$ is a function on the phase space and the ‘‘irrelevant terms’’ are conserved separately (in other words, to make the integral in (V.25) local we can use the equations of the motion, thus both forms of \tilde{G}_Y can differ by other integrals of the motion). This local integral of the motion generates a canonical symmetry transformation,

$$\tilde{\delta}x^\mu = \epsilon\{x^\mu, \tilde{G}_Y\}, \quad \tilde{\delta}p_\mu = \epsilon\{p_\mu, \tilde{G}_Y\}. \quad (\text{V.27})$$

One has also

$$\{\tilde{G}_Y, \tilde{G}_{Y_1}\} = \{G_Y, G_{Y_1}\} + (\dots)\mathcal{H}_a. \quad (\text{V.28})$$

We conclude that: (i) on the submanifold $\mathcal{H}_a = 0$ the algebras generated by G_Y 's and \tilde{G}_Y 's are isomorphic and they are (anti) isomorphic to the conformal algebra (ii) the canonical transformation on the phase space generated by G_Y 's and \tilde{G}_Y 's are in general different, even when restricted to the submanifold $\mathcal{H} = 0$; (iii) \tilde{G}_Y leaves invariant all submanifolds

$\mathcal{H}_a = -\frac{m^2}{2}$ while G_Y only leaves invariant the submanifold $\mathcal{H}_a = 0$ (except when Y is a genuine Killing field).

In view of the above discussion the crucial point is the possibility of localization, cf. (V.26). The more we know about the solution of the equation of the motion the more likely is that we can resolve the localization problem. When the solutions are known explicitly, the localization problem can be solved immediately ; in such a case all integrals of the motion are explicitly known so our procedure is then not a very useful one.

However, there exists another possibility: due to the special form of the conformal factor only some partial information about the solutions is necessary – and such information may be available, because (for example) of other conservation laws. Then we may construct new conservation laws by combining the conformal transformations with the already known conservations laws. This is what happens for pp-waves. Namely, in this case Ω is simply,

$$\Omega_Y(x, p) = \frac{2}{p_v^2} f^\mu p_\mu, \quad (\text{V.29})$$

where $f = f(x, p)$ is defined by (V.6) with the replacement $m^2 \rightarrow -2\mathcal{H}_a$ (cf. eqn. (V.25)), and it gives an ordinary integral of the motion \tilde{G}_Y .

The non-point, canonical symmetries generated by the \tilde{G}_Y 's can be put in the Lagrangian form if the momenta are replaced by the appropriate combinations of velocities; however, then the infinitesimal transformations involve, in general, also velocities (in contrast to Noether point symmetries).

To conclude this section let us discuss whether any new information is carried by the “conformal” charges. For any genuine Killing vector one obtains an integral of the motion. Thus for sufficiently symmetric space-time the number of independent integrals of the motion associated with the Killing vectors can attain the maximal value which, for four-dimensional manifolds, equals 7. Then the dynamics governed by \mathcal{H}_a is superintegrable.

Any additional integral of the motion is a function of those basic ones. Such a situation takes place for the flat Minkowski space-time : we have 10 Killing vectors corresponding to the Poincaré symmetry. The components of four-momenta p_μ and the boosts, M_{0a} , $a = 1, 2, 3$ form 7 independent integrals. One can verify by explicit computations that the charges associated with all conformal generators (as obtained in the present paper) are rationally expressible in terms of them (see sec. VIA).

However, in the case of general pp-waves the situation is different. It turns out that (see

[45]) apart from some special cases, there is only one Killing vector ∂_v for generic pp-waves (giving only one integral of the motion, p_v). On the other hand, some classes of pp-waves admit three proper conformal fields [22]. Moreover, in sec. IV C we showed that, even for the Minkowski spacetime, there are electrodynamic backgrounds which break the Poincaré symmetry but are preserved by conformal fields. The resulting charges are not functions of the Hamiltonian and p_v only and thus provide explicit examples where one obtains new information about the geodesics equation from conformal symmetry. We believe that this is the main reason for which the formalism considered in this paper may be really useful.

To conclude, let us note that eqs. (V.24) and (V.28) imply that the Poisson bracket of two localisable charges gives again a localisable one. Therefore, starting from one such charge and taking its Poisson bracket with the charge generated by a Killing vector (which is thus localisable) one produces another localisable charge. This process can be continued which implies that the structure of the conformal algebra plays an important role for localization.

VI. EXAMPLES

A. A free relativistic particle

For a free particle in Minkowski space all Christoffel symbols vanish and the equations of the motion become

$$\ddot{x}^\alpha = \left(\frac{d}{d\tau} \ln(\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}) \right) \dot{x}^\alpha. \quad (\text{VI.1})$$

Equivalently, in terms of the canonical momenta,

$$\frac{dp_\alpha}{d\tau} = 0 \quad \text{where} \quad p_\alpha \equiv \frac{\partial L}{\partial \dot{x}^\alpha} = \frac{mg_{\alpha\beta}\dot{x}^\beta}{\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}. \quad (\text{VI.2})$$

Eqn. (VI.1) is integrated as $\dot{x}^\alpha = Ng^{\alpha\beta}p_\beta$, $N = m^{-1}\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$. Using light-cone coordinates this becomes $\dot{x}^i = Np_i$, $\dot{u} = Np_v$, $\dot{v} = Np_u$, where p_i, p_u, p_v are all constants. Skipping the isometries, we consider 5 proper conformal transformations,

$$Y_D = 2u\partial_u + x^i\partial^i \quad (\mathcal{L}_{Y_D}g)_{\mu\nu} = 2g_{\mu\nu}, \quad (\text{VI.3a})$$

$$Y_K = u^2\partial_u + ux^i\partial^i - \frac{\mathbf{x}^2}{2}\partial_v \quad (\mathcal{L}_{Y_K}g)_{\mu\nu} = 2u g_{\mu\nu}, \quad (\text{VI.3b})$$

$$Y_{C1} = \frac{\mathbf{x}^2}{2}\partial_u - vx^i\partial^i - v^2\partial_v \quad (\mathcal{L}_{Y_{C1}}g)_{\mu\nu} = -2v g_{\mu\nu}, \quad (\text{VI.3c})$$

$$Y_{C2}^i = x^i u \partial_u - \left(\frac{\mathbf{x}^2}{2} + uv \right) \partial^i + x^i (x^j \partial^j) + x^i v \partial_v \quad (\mathcal{L}_{Y_{C2}^i}g)_{\mu\nu} = 2x^i g_{\mu\nu}. \quad (\text{VI.3d})$$

to which (II.9) associates the seemingly non-local conserved charges

$$I_j = Y_j^\alpha p_\alpha + m^2 \int \omega_j(x(\tau)) N(\tau) d\tilde{\tau}. \quad (\text{VI.4})$$

The $Y^\alpha p_\alpha$ terms can be written out explicitly using (VI.3) and (VI.2). The clue to determine the integral term is the property that each conformal factor depends on one coordinate only. The integral can therefore be evaluated and yields simple local expressions,

1. For the field Y_D in (VI.3a) $\omega = 1$; eliminating N by $\dot{u} = Np_v$ yields ³,

$$I_{D1} = 2up_u + x^i p_i + \frac{m^2}{p_v} u + \text{const.} \quad (\text{VI.5})$$

2. For an expansion Y_K in (VI.3b) we have $\omega = u$, and we may choose $\dot{u} = Np_v$; the integration yields

$$\mathcal{J}_K = m^2 \omega \int N(\tau) d\tau = \frac{m^2}{2p_v} u^2 + \text{const.} \quad (\text{VI.6})$$

3. Similarly for Y_{C1} in (VI.3c) $\omega = -v$ and we may choose $\dot{v} = Np_u$ to get

$$\mathcal{J}_{C1} = m^2 \int \omega(v(\tau)) N(\tau) d\tau = -\frac{m^2}{2p_u} v^2 + \text{const.} \quad (\text{VI.7})$$

4. At last for Y_{C2} in (VI.3d) we get, by choosing $N = \dot{x}^i/p_i$,

$$\mathcal{J}_{C2} = m^2 \int^\tau \omega(\tilde{\tau}) N(\tilde{\tau}) d\tilde{\tau} = \frac{m^2}{2p_i} x^{i2} + \text{const.} \quad (\text{VI.8})$$

All these expressions are *local*. The non-trivial distorted conformal vectors are the $\Upsilon^\alpha = Y^\alpha + \frac{m^2}{p_v^2} f^\alpha$ with

$$f_D = u\partial_v \quad (\text{VI.9})$$

$$f_K = \frac{u^2}{2}\partial_v \quad (\text{VI.10})$$

$$f_{C1} = \left[-\frac{1}{4} \frac{m^2}{p_v^2} u^2 - uv + \frac{1}{4} (x^2 + y^2) \right] \partial_v - \frac{u}{2} x^i \partial_i \quad (\text{VI.11})$$

$$f_{C2}^i = ux^i \partial_v - \frac{u^2}{2} \partial^i. \quad (\text{VI.12})$$

Note that the mass appears in f_{C1} only. When $m \neq 0$, the non-Killing Υ vectors do *not* yield a closed algebra in general : for example, $[\Upsilon_K, \Upsilon_{C2}^i] = \frac{m^2}{p_v^2} [u^2 x^i \partial_v - u^3 \partial^i]$.

³ Choosing instead $\dot{v} = Np_u$ also allowed $I_{D2} = 2up_u + x^i p_i + \frac{m^2}{p_u} v + \text{const.}$ These expressions are equivalent because $\dot{u}/p_v = \dot{v}/p_u = N$.

Moreover, the “mass distorted” Lie derivative formula (V.19) can be confirmed by means of the vector field Y_{C1} and its distorted conformal factor $\omega_m = -v - \frac{m^2 u}{2p^2}$.

Now, following sec. VC, we rewrite the above charges in the form (V.26) (see also (V.29)). Then the \tilde{G} 's are ordinary integrals of the motion for the relativistic particle in the affine parametrisation. Since such a system is superintegrable the charges \tilde{G} 's should be expressible in terms of the basic charges (related to the Poincaré symmetry, see the discussion in sec. VC). In fact, after straightforward but tedious computations one finds that all conformal generators are rational functions of the 7 independent integrals p_μ and the boots M_{0a} , $a = 1, 2, 3$; e.g. for the conformal generator K we obtain

$$\tilde{G}_K = \frac{1}{2\sqrt{2}(p^0)^2(p^0 - p^3)} ((p^3 - p^0)M^{i0} - p^i M^{30})^2 \quad (\text{VI.13})$$

All these integrals of the motion generate symmetries which are not point transformations. Thus on the Lagrangian level the infinitesimal transformations contain velocities; e.g. for the charge (VI.13) the corresponding transformation of the configuration space is

$$\delta u = 0, \quad \delta v = \frac{1}{2}(-\mathbf{x}^2 + \frac{u^2}{u'}\mathbf{x}'^2), \quad \delta \mathbf{x} = u\mathbf{x} - \frac{u^2}{u'}\mathbf{x}' \quad (\text{VI.14})$$

and it leaves the equation $x^{\mu''} = 0$ invariant.

B. The conformally flat isotropic oscillator

Choosing

$$H(u, x, y) = x^2 + y^2 \quad (\text{VI.15})$$

in (III.1) yields a conformally flat (not Einstein-vacuum) metric. In “Bargmann” terms the latter describes a two-dimensional isotropic inverted harmonic oscillator ; an attractive oscillator would be obtained by changing the overall sign of H in (VI.15).

As the properties of this case have been studied by many authors, here we merely list the principal results. The metric admits fifteen conformal Killing vectors : the well-known seven

Killing vectors Y_J , $J = 1, \dots, 7$ are completed by eight truly conformal ($\omega_J \neq 0$) generators

$$Y_8 = x\partial_x + y\partial_y + 2v\partial_v \quad (\text{VI.16a})$$

$$Y_9 = e^{2u}\partial_u + e^{2u}x\partial_x + e^{2u}y\partial_y - e^{2u}(x^2 + y^2)\partial_v \quad (\text{VI.16b})$$

$$Y_{10} = -e^{-2u}\partial_u + e^{-2u}x\partial_x + e^{-2u}y\partial_y + e^{-2u}(x^2 + y^2)\partial_v \quad (\text{VI.16c})$$

$$Y_{11} = -(x^2 + y^2)\partial_u + 2vx\partial_x + 2vy\partial_y + \left[2v^2 + \frac{1}{2}(x^2 + y^2)^2\right]\partial_v \quad (\text{VI.16d})$$

$$Y_{12} = e^u x\partial_u - \frac{e^u}{2}(2v - x^2 + y^2)\partial_x + e^u xy\partial_y + \frac{e^u x}{2}(2v - x^2 - y^2)\partial_v \quad (\text{VI.16e})$$

$$Y_{13} = -e^{-u}x\partial_u + \frac{e^{-u}}{2}(2v + x^2 - y^2)\partial_x + e^{-u}xy\partial_y + \frac{e^{-u}x}{2}(2v + x^2 + y^2)\partial_v \quad (\text{VI.16f})$$

$$Y_{14} = e^u y\partial_u + e^u xy\partial_x - \frac{e^u}{2}(2v + x^2 - y^2)\partial_y + \frac{e^u y}{2}(2v - x^2 - y^2)\partial_v \quad (\text{VI.16g})$$

$$Y_{15} = -e^{-u}y\partial_u + e^{-u}xy\partial_x + \frac{e^{-u}}{2}(2v - x^2 + y^2)\partial_y + \frac{e^{-u}y}{2}(2v + x^2 + y^2)\partial_v \quad (\text{VI.16h})$$

with conformal factors

$$\begin{aligned} \omega_8 &= 1, & \omega_9 &= e^{2u}, & \omega_{10} &= e^{-2u}, \\ \omega_{11} &= 2v & \omega_{12} &= e^u x, & \omega_{13} &= e^{-u} x, & \omega_{14} &= e^u y, & \omega_{15} &= e^{-u} y, \end{aligned} \quad (\text{VI.17})$$

yielding the associated integrals of the motion

$$I_8 = \frac{m^2}{p_v^2} u + \frac{x\dot{x}}{\dot{u}} + \frac{y\dot{y}}{\dot{u}} + 2v \quad (\text{VI.18a})$$

$$I_9 = \frac{1}{2} \frac{m^2}{p_v^2} e^{2u} + \frac{e^{2u}\dot{v}}{\dot{u}} + \frac{e^{2u}x\dot{x}}{\dot{u}} + \frac{e^{2u}y\dot{y}}{\dot{u}} \quad (\text{VI.18b})$$

$$I_{10} = -\frac{m^2}{2p_v^2} e^{-2u} - \frac{e^{-2u}\dot{v}}{\dot{u}} + \frac{e^{-2u}x\dot{x}}{\dot{u}} + \frac{e^{-2u}y\dot{y}}{\dot{u}} \quad (\text{VI.18c})$$

$$I_{11} = \frac{m^2}{2p_v^2} \left(\frac{m^2}{p_v^2} u^2 + 4uv - x^2 - y^2 \right) + \left(\frac{m^2}{p_v^2} ux + 2vx \right) \frac{\dot{x}}{\dot{u}} + \left(\frac{m^2}{p_v^2} uy + 2vy \right) \frac{\dot{y}}{\dot{u}} \\ - \frac{(x^2 + y^2)\dot{v}}{\dot{u}} + \left(2v^2 - \frac{1}{2}(x^2 + y^2)^2 \right) \quad (\text{VI.18d})$$

$$I_{12} = \frac{m^2 e^u}{4p_v^2} \left((2u+1)x - (2u-1) \frac{\dot{x}}{\dot{u}} \right) + e^u x \frac{\dot{v}}{\dot{u}} \\ + \frac{e^u x}{2} (2v + x^2 + y^2) - \frac{e^u}{2} (2v - x^2 + y^2) \frac{\dot{x}}{\dot{u}} + e^u xy \frac{\dot{y}}{\dot{u}} \quad (\text{VI.18e})$$

$$I_{13} = \frac{m^2 e^{-u}}{4p_v^2} \left((2u-1)x + (2u+1) \frac{\dot{x}}{\dot{u}} \right) - e^{-u} x \frac{\dot{v}}{\dot{u}} \\ + \frac{e^{-u} x}{2} (2v - x^2 - y^2) + \frac{e^{-u}}{2} (2v + x^2 - y^2) \frac{\dot{x}}{\dot{u}} + e^{-u} xy \frac{\dot{y}}{\dot{u}} \quad (\text{VI.18f})$$

$$I_{14} = \frac{m^2 e^u}{4p_v^2} \left((2u+1)y - (2u-1) \frac{\dot{y}}{\dot{u}} \right) + e^u y \frac{\dot{v}}{\dot{u}} \\ + \frac{e^u y}{2} (2v + x^2 + y^2) - \frac{e^u}{2} (2v + x^2 - y^2) \frac{\dot{y}}{\dot{u}} + e^u xy \frac{\dot{x}}{\dot{u}} \quad (\text{VI.18g})$$

$$I_{15} = \frac{m^2 e^{-u}}{4p_v^2} \left((2u-1)y + (2u+1) \frac{\dot{y}}{\dot{u}} \right) - e^{-u} y \frac{\dot{v}}{\dot{u}} \\ + \frac{e^{-u} y}{2} (2v - x^2 - y^2) + \frac{e^{-u}}{2} (2v - x^2 + y^2) \frac{\dot{y}}{\dot{u}} + e^{-u} xy \frac{\dot{x}}{\dot{u}} \quad (\text{VI.18h})$$

All integrands are total derivatives and therefore all quantities are local.

C. A conformally non-flat vacuum pp wave

Now we turn to an u -dependent vacuum pp-wave metric, (III.1) with

$$H(u, x, y) = \frac{1}{u^4} (x^2 - y^2). \quad (\text{VI.19})$$

The regularized version of this metric has been considered before [7, 27, 28] ; here we revisit these results using our new framework.

Solving the conformal Killing equations $\mathcal{L}_Y g_{\alpha\beta} = 2\omega g_{\alpha\beta}$ yields five Killing fields ($\omega_i = 0$, $i = 1, \dots, 5$), we list them for completeness,

$$\begin{aligned} Y_1 &= \partial_v, & Y_2 &= \frac{e^{1/u}(u-1)x}{u}\partial_v - e^{1/u}u\partial_x, & Y_3 &= \frac{e^{-1/u}(u+1)x}{u}\partial_v - e^{-1/u}u\partial_x, \\ Y_4 &= y \left[\frac{1}{u} \sin\left(\frac{1}{u}\right) + \cos\left(\frac{1}{u}\right) \right] \partial_v - u \cos\left(\frac{1}{u}\right) \partial_y, \\ Y_5 &= y \left[\sin\left(\frac{1}{u}\right) - \frac{1}{u} \cos\left(\frac{1}{u}\right) \right] \partial_v - u \sin\left(\frac{1}{u}\right) \partial_y. \end{aligned} \quad (\text{VI.20})$$

The proper conformal fields are the homothety, Y_6 and Y_7 in (III.12) resp. (VI.3b), with conformal factors $\omega_6 = 1$ and $\omega_7 = u$. General theorems say that the maximal number of conformal vectors of a non-conformally-flat pp wave is 7 [45] – a number which is attained in this case.

The Lagrangian describing a massive relativistic particle moving in such space-time is given by (II.7) with Euler-Lagrange equations (III.2) and H defined by (VI.19). The trajectories in the transverse space are plotted in Fig.1.

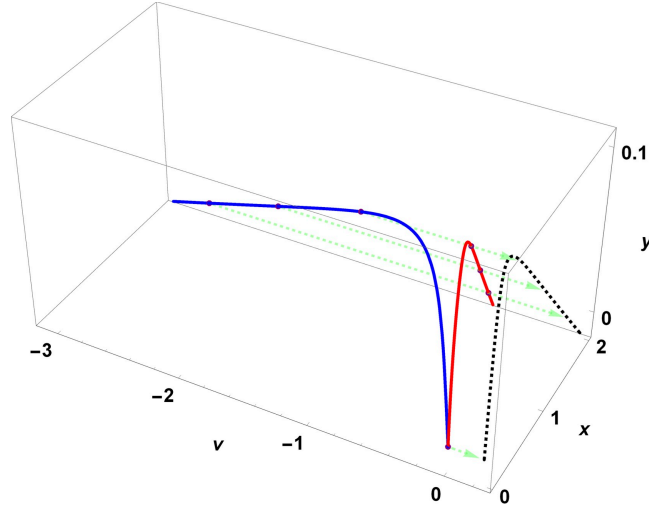


FIG. 1: The massive $m \neq 0$ (in blue) and massless $m = 0$ (in red) trajectories project on the transverse plane onto the same (dotted) curve. Their v -coordinates differ by $(m/p_v)^2 u/2$.

For the homothety Y_6 and the proper conformal Killing vector Y_7 eqn. (II.9) with $\phi = 0$ yields the manifestly local conserved charges

$$I_6 = \frac{p_v}{\dot{u}} (x\dot{x} + y\dot{y}) + 2p_v v + \frac{m^2}{p_v} u + \text{const.} \quad (\text{VI.21a})$$

$$I_7 = \frac{p_v u^2 \dot{v}}{\dot{u}} + \frac{p_v u (x\dot{x} + y\dot{y})}{\dot{u}} - \frac{p_v}{2} (x^2 + y^2) - \frac{p_v}{u^2} (y^2 - x^2) + \frac{m^2}{2p_v} u^2 + \text{const.} \quad (\text{VI.21b})$$

The mass-dependent terms can actually be eliminated. To see this we observe that I_6 is a combination of E_N , E_i (given $N = \frac{\dot{u}}{p_v}$). In particular,

$$-\frac{dI_6}{d\tau} = 2\frac{\dot{u}}{p_v}E_N + xE_x + yE_y. \quad (\text{VI.22})$$

One can obtain another integral of the motion I_0 which does not depend on the mass by subtracting from I_6 (or from I_7) the integrated equation (III.4) for \dot{v} :

$$I_0 = 2v + \frac{m^2}{p_v^2}u + \int \left[\frac{1}{\dot{u}}(\dot{x}^2 + \dot{y}^2) + \frac{\dot{u}}{u^4}(x^2 - y^2) \right] d\tau. \quad (\text{VI.23})$$

Any linear combination of two integrals of the motion is again an integral of the motion; in particular we find

$$\tilde{I}_6 = \frac{1}{p_v}I_6 - I_0 = \frac{1}{\dot{u}}(x\dot{x} + y\dot{y}) - \int \left[\frac{1}{\dot{u}}(\dot{x}^2 + \dot{y}^2) + \frac{\dot{u}}{u^4}(x^2 - y^2) \right] d\tau. \quad (\text{VI.24})$$

\tilde{I}_6 does not depend on the mass and it is now - since we eliminated E_N - an integral for the E_i equations, $\frac{d\tilde{I}_6}{d\tau} = yE_y - xE_x$.

Following the same procedure applied to I_7 we end up with an expression which again does not depend on the mass : we get an integral of the motion for the (x, y) equations alone, which is *exactly the same as in the massless case*. The mass appears in the constraint equation and through it affects only the v variable. Integrals of the motion which do not contain the v variable are independent of the mass.

Finally, let us see what one gets from the modified Noether procedure. The symmetry generators (V.10) are

$$\begin{aligned} \Upsilon_i &= Y_i, \quad i = 1, \dots, 5 \\ \Upsilon_6 &= Y_6 + \frac{m^2}{p_v^2}u \partial_v \\ \Upsilon_7 &= Y_7 + \frac{m^2}{2p_v^2}u^2 \partial_v. \end{aligned} \quad (\text{VI.25})$$

The Killing vector fields Y_i , $i = 1, \dots, 5$, yield the Noether symmetries, while the modification provides us with additional mass-dependent ‘‘corrections’’ for the conformal Killing vectors.

Obviously $Q_i = I_i$ for $i = 1, \dots, 5$ and for the last two Q 's (V.9) we obtain

$$Q_6 = \frac{1}{N}(x\dot{x} + y\dot{y}) + 2v\frac{\dot{u}}{N} + \frac{m^2}{p_v^2}u\frac{\dot{u}}{N} = I_6, \quad (\text{VI.26a})$$

$$Q_7 = \frac{u^2\dot{v}}{N} + \frac{u(x\dot{x} + y\dot{y})}{N} - \frac{((u^2 - 2)x^2 + (u^2 + 2)y^2)\dot{u}}{2Nu^2} + \frac{m^2}{p_v^2}u^2\frac{\dot{u}}{2N} = I_7. \quad (\text{VI.26b})$$

Lastly we note that the ‘‘distorted’’ Killing vectors (VI.25) and the associated conserved charges (VI.26) satisfy (V.19) and (V.18), respectively.

VII. CONCLUSION

In this work we investigated the conserved charges associated with conformal Killing fields in curved space-times possibly equipped also with an electromagnetic background preserved by those Killing fields. We put special emphasis on massive particles – those which are mostly considered in the Memory Effect [10]. The associated conserved quantity in (II.3) involves an integral term, eqn. (I.1), which requires integration along the trajectory and could therefore be non-local. It is only for a special parametrisation that this term become local general. However, such conceptual and calculational difficulties are absent in pp-wave space-times : the integral term can be calculated analytically and becomes local in an arbitrary parametrisation, as implied by (V.8).

pp-wave space-times play a role for the Eisenhart-Duval lift [12–14, 16] of 2-dimensional classical dynamics. Analysing the meaning of the charges in this context, we have shown that after expressing them in terms of transversal coordinates, the term with m reduces to a constant. This is consistent with the observation that both massive and massless geodesic motion are lifts of the same underlying classical dynamics.

Moreover, considering conformally related (and consequently physically inequivalent) pp-wave metrics, we have shown that the conformal charges for massive geodesics coincide. Explicit examples allow us to give more clear interpretation of some charges corresponding to proper conformal fields.

Next, we constructed a family of pp-waves (which include the Minkowski space-time) and an independent family of electromagnetic backgrounds which are preserved by a suitable conformal field. The explicit form of the corresponding charges was given. We gave the example of electromagnetic backgrounds for which the conformal symmetry yields a new integral of the motion.

In the usual approach the Killing vectors can be identified with Noether point symmetries (and consequently give Noether charges which are linear in the momenta); however, for massive particles, the conformal vectors do *not* define symmetries [37] in general.

In eqn (V.10) of sec.V we introduced “distorted”, non-point transformations, which are analogous to dynamical symmetries and related them to local conformal charges. First, we rewrote the local charges in an “almost Noetherian” form (using the parametrisation invariant approach); although the distorted field contains momenta and isn’t formally a

point symmetry, fixing $p_v = \text{const.}$ allows us to interpret it as such.

Moreover, in the context of the charges we obtained, we modified appropriately the Noether procedure by fixing the momentum by a supplementary condition); and discussed its geometric meaning.

Next, we analysed the charges as associated with the symmetries of the canonical Hamilton equations (using the Hamiltonian approach with affine parametrisation). In view of these considerations the conformal Killing fields, together with an appropriate distortion, generate non-point symmetry transformations and induce velocity-dependent transformations in the configuration space.

At last we discussed the possibilities of localization of conformal charges and their relevance in integrability of geodesics equations and presented some further examples.

The process that we followed can be extended in other configurations apart from pp-waves. The results we obtained fit into various recent studies of the relations between conformal symmetries and integrability problems, shedding new light at these issues.

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- [1] M. Henkel, “Local Scale Invariance and Strongly Anisotropic Equilibrium Critical Systems,” *Phys Rev. Lett.* **78** (1997), 1940 ; “Phenomenology of local scale invariance: from conformal invariance to dynamical scaling,” *Nucl. Phys. B* **641** (2002), 405;
 - [2] T. Igata, “Scale invariance and constants of the motion,” *PTEP* **2018** (2018) no.6, 063E01 doi:10.1093/ptep/pty060 [arXiv:1804.03369 [hep-th]].
 - [3] D. Maughan and C. Torre, “Affine symmetry, geodesics, and homogeneous spacetimes,” *Gen. Rel. Grav.* **50** (2018) no.8, 102 doi:10.1007/s10714-018-2422-0 [arXiv:1807.08180 [gr-qc]].
 - [4] P.-M. Zhang, M. Elbistan, P. A. Horvathy and P. Kosinski, “A generalized Noether theorem for scaling symmetry,” *Eur. Phys. J. Plus* (2020) 135:223 doi.org/10.1140/epjp/s13360-020-

- 00247-5 [arXiv:1903.05070 [math-ph]].
- [5] A. I. Harte, “Optics in a nonlinear gravitational plane wave,” *Class. Quant. Grav.* **32** (2015) no.17, 175017 doi:10.1088/0264-9381/32/17/175017 [arXiv:1502.03658 [gr-qc]].
- [6] M. Cariglia, G.W. Gibbons, J.-W. van Holten, P.A. Horvathy, P.-M. Zhang “Conformal Killing Tensors and covariant Hamiltonian Dynamics,” *J. Math. Phys.* **55** (2014) 122702 ics
- [7] P.-M. Zhang, M. Cariglia, M. Elbistan and P. A. Horvathy, “Scaling and conformal symmetries for plane gravitational waves,” *J. Math. Phys.* **61**, 022502 (2020) doi:10.1063/1.5136078, 10.1063/1.513607 [arXiv:1905.08661 [gr-qc]];
- [8] Ya. B. Zel’dovich and A. G. Polnarev, “Radiation of gravitational waves by a cluster of superdense stars,” *Astron. Zh.* **51**, 30 (1974) [*Sov. Astron.* **18** 17 (1974)]; V B Braginsky and L P Grishchuk, “Kinematic resonance and the memory effect in free mass gravitational antennas,” *Zh. Eksp. Teor. Fiz.* **89** 744-750 (1985) [*Sov. Phys. JETP* **62**, 427 (1985)]; V B Braginsky and K S Thorne, “Gravitational-wave burst with memory and experimental prospects,” *Nature (London)* **327** 123 (1987). For further details and an extensive list of recent references, see, e.g., [10].
- [9] T. Heinzl, A. Ilderton, “Exact classical and quantum dynamics in background electromagnetic fields,” *Phys. Rev. Lett.* **118** (2017) 113202; T. Heinzl, A. Ilderton, “Superintegrable relativistic systems in spacetime-dependent background fields,” *J. Phys. A: Math. Theor.* **50** (2017) 345204; A. Ilderton, “Screw-symmetric gravitational waves: a double copy of the vortex,” *Phys. Lett. B* **782** (2018) 22.
- [10] P.-M. Zhang, C. Duval, G. W. Gibbons and P. A. Horvathy, “The Memory Effect for Plane Gravitational Waves,” *Phys. Lett. B* **772** (2017) 743. doi:10.1016/j.physletb.2017.07.050 [arXiv:1704.05997 [gr-qc]] ; P.-M. Zhang, C. Duval, G. W. Gibbons and P. A. Horvathy, “Soft gravitons and the memory effect for plane gravitational waves,” *Phys. Rev. D* **96** (2017) no.6, 064013 doi:10.1103/PhysRevD.96.064013. [arXiv:1705.01378 [gr-qc]].
- [11] N. Dimakis, P. Terzis, T. Christodoulakis, “Integrability of geodesic motions in curved manifolds through non-local conserved charges,” *Phys. Rev. D* **99** (2019) 104061
- [12] L. P. Eisenhart, “Dynamical trajectories and geodesics”, *Annals Math.* **30** 591-606 (1928).
- [13] C. Duval, G. Burdet, H. Kunzle, M. Perrin, “Bargmann structures and Newton-Cartan theory,” *Phys. Rev. D* **31** (1985) 1841
- [14] C. Duval, G.W. Gibbons, P. Horvathy, “Celestial mechanics, conformal structures and gravi-

- tational waves,” *Phys. Rev.* **D43** (1991) 3907. [hep-th/0512188].
- [15] X. Bekaert and K. Morand, “Embedding nonrelativistic physics inside a gravitational wave,” *Phys. Rev. D* **88** (2013) no.6, 063008 doi:10.1103/PhysRevD.88.063008 [arXiv:1307.6263 [hep-th]];
- [16] M. Cariglia, C. Duval, G. W. Gibbons and P. A. Horvathy, “Eisenhart lifts and symmetries of time-dependent systems,” *Annals Phys.* **373** (2016) 631 doi:10.1016/j.aop.2016.07.033 [arXiv:1605.01932 [hep-th]].
- [17] P. A. M. Dirac, *Lectures on Quantum Mechanics*, Yeshiva University Press, New York (1964)
- [18] K. Sundermeyer, *Constrained Dynamics*, Springer Verlag, Berlin, Heidelberg, New York (1982)
- [19] E. Noether, “Invariante Variationsproblem”, *Gott. Nachr.* **1918** (1918) 235; [English translation by M. A. Tavel: E. Noether, “Invariant Variation Problems” *Transp. Theory Statist. Phys.* **1** (1971) 186]
- [20] K. V. Kuchař, “Conditional Symmetries In Parametrized Field Theories” *J. Math. Phys.* **23**, (1982) 1647
- [21] L. Brink, P. Di Vecchia, P. Howe, “A Lagrangian formulation of the classical and quantum dynamics of spinning particles,” *Nucl. Phys.* **B 118** (1977) 76.
- [22] A. Keane and B. Tupper, “Conformal symmetry classes for pp-wave space-times,” *Class. Quant. Grav.* **21** (2004) 2037
- [23] R. Maartens, S. Maharaj, “Conformal symmetries of pp-waves,” *Class. Quant. Grav.* **8** (1991) 503
- [24] M. Perrin, G. Burdet and C. Duval, “Chrono-projective Invariance of the Five-dimensional Schrödinger Formalism,” *Class. Quant. Grav.* **3** (1986) 461. doi:10.1088/0264-9381/3/3/020
- [25] P.-M. Zhang, M. Cariglia, M. Elbistan, G. W. Gibbons and P. A. Horvathy, ““Kepler Harmonies” and conformal symmetries,” *Phys. Lett. B* **792** (2019) 324 doi:10.1016/j.physletb.2019.03.057 [arXiv:1903.01436 [gr-qc]];
- [26] W. Kühnel, H.-B. Rademacher, “Conformal geometry of gravitational plane waves,” *Geom. Dedicata* **109** (2004) 175
- [27] K. Andrzejewski, S. Prencel, “Memory effect, conformal symmetry and gravitational plane waves,” *Phys. Lett. B* **782** (2018) 421
- [28] K. Andrzejewski, S. Prencel, “Niederer’s transformation, time-dependent oscillators and polarized gravitational waves,” *Class. Quantum Grav.* **36** 155008 (2019)

- [29] K. Andrzejewski, S. Prencel, "From polarized gravitational waves to analytically solvable electromagnetic beams," *Phys. Rev. D* **100** (2019) 045006
- [30] H. Lewis, "Classical and Quantum Systems with Time-Dependent Harmonic-Oscillator-Type Hamiltonians," *Phys. Rev. Lett.* **18** (1967) 510
- [31] H. Lewis, W. Riesenfeld, "Class of Exact Invariants for Classical and Quantum Time-Dependent Harmonic Oscillators," *J. Math. Phys.* **10** (1969) 1458
- [32] U. Niederer, "The maximal kinematical invariance group of the harmonic oscillator," *Helv. Phys. Acta* **46** (1973) 191
- [33] R. Monteiro, D. O'Connell, C. White, "Black holes and the double copy," *JHEP* **12** (2014) 056
- [34] T. Christodoulakis, N. Dimakis and Petros A. Terzis, "Lie point and variational symmetries in minisuperspace Einstein gravity," *J. Phys. A: Math. Theor.* **47**, (2014) 095202
- [35] M. Tsamparlis and A. Paliathanasis, "Lie and Noether symmetries of geodesic equations and collineations" *Gen. Rel. Grav.* **42** (2010) 2957
- [36] M. Tsamparlis and A. Paliathanasis, "The geometric nature of Lie and Noether symmetries" *Gen. Rel. Grav.* **43** (2011) 1861
- [37] T. Feroze and I. Hussain, "Noether symmetries and conserved quantities for spaces with a section of zero curvature", *J. Geom. and Phys.* **61** (2011) 658; I. Hussain, "A note on Noether symmetries and conformal Killing vectors", *Gen. Relativ. Gravit.* **42** (2010) 1791
- [38] P. J. Olver, *Applications of Lie Groups to Differential Equations*, 2nd edn, Berlin, Springer, (2000).
- [39] G. W. Bluman and S. Kumei, *Symmetries and Differential Equations*, Springer-Verlag, New York, Berlin, Heidelberg (1989)
- [40] R. L. Anderson and N. H. Ibragimov, *Lie-Bäcklund Transformations in Applications*, SIAM Studies in Applied Mathematics, SIAM, Philadelphia (1979)
- [41] S. Hojman, L. Nuñez, A. Patiño and H. Rago, "Symmetries and conserved quantities in geodesic motion" *J. Math. Phys.* **27** (1986) 281
- [42] P. Jizba and J. M. Pons, "Revisiting the gauge principle: enforcing constants of the motion as constraints", *J. Phys. A: Math. Theor.* **43**, (2010) 205202
- [43] G. E. Prince and M. Crampin, "Projective Differential Geometry and Geodesic Conservation Laws in General Relativity. I: Projective Actions, *General. Relativ. and Grav.* **16** (1984) 921

- [44] G. H. Katzin and J. Levine, “Geodesic first integrals with explicit pathparameter dependence in Riemannian spacetimes”, *J. Math. Phys.* **22**, (1981) 1878
- [45] D. Kramer, H. Stephani, M. McCallum, E. Herlt, “Exact solutions of Einstein’s field equations,” Cambridge Univ. Press (1980).
- [46] A. Patiño, “Symmetries and constants of the motion of geodesic equations” *Can. J. Phys.* **67** (1989) 485
- [47] P. A. Damianou and C. Sophocleous, “Symmetries of Hamiltonian systems with two degrees of freedom,” *J. Math. Phys.* **40** (1999) 210